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Partitions – A Survey*

Hansraj Gupta**

(September 19, 1969)

A historical survey of some aspects of the theory of partitions is given here.

Key words: Congruences; generating functions; identities; partitions; recurrence formulas.

1. Introduction

1.1. Unrestricted Partitions

As far back as 1669, Leibnitz asked John Bernoulli, if he had investigated the number of ways in which a given number can be expressed as a sum of two or more integers. This was necessarily the problem of partitioning a given number into a specified number of parts.

A partition of a positive integer n, is a mode of expressing it as a sum of one or more positive integers, the order in which the summands occur being irrelevant. The parts are usually arranged according to size, one way or the other. Thus, 5 can be expressed as a sum in the following seven ways:

$$5; 1+4; 2+3; 1+2+2; 1+1+3; 1+1+1+2; \text{ and } 1+1+1+1+1.$$

These are all the partitions of 5. Since there is no restriction of any kind on the number or size of parts, we call these the unrestricted partitions of 5. In all that follows, the number of unrestricted partitions of any given positive integer n will be denoted by $p(n)$ and we shall take $p(0)=1$.

In this survey, we shall give a short resumé of the advances that have been made in recent years, in dealing with problems on partitions of integers. The bibliography includes papers on partitions of vectors also but we do not propose to deal with the topic here.

1.2. Restricted Partitions

Given a set $$A=\{a_1, a_2, a_3, \ldots\ldots\},$$

where the a's are positive integers, one may be required to find the number of ways in which a given integer can be expressed as a sum of the a's, repetitions being allowed or not. This will be an example of restricted partitions. As other examples, we can mention: (1) Partitions into distinct parts; (2) Partitions into odd parts; (3) Partitions into a specified number of parts; and (4) Partitions into a specified number of distinct parts. In fact, the restrictions that can be placed on the size or on the number of parts or both, are too many to enumerate. Very recently, I had to consider, for example, the number $g(n, m, h, k)$ of partitions of n into exactly k summands, each $\leqslant m$, just h (and any h) of the positive integers $\leqslant m$ being used as summands, in any partition. Among several interesting results, I found that

$$\sum_{n=1}^{\infty} g(n, m, h, k) = \binom{k-1}{h-1}\binom{m}{h}.$$

*An invited paper. This survey is based on an address given by the writer at a conference of the Indian Mathematical Society held at Madras in December 1963. It has been brought to date with the help of Professor M. S. Cheema of the University of Arizona, Tucson.

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We shall denote by $p(n, k)$ the number of partitions of n into exactly k summands; by $q(n)$ the number of partitions of n into distinct parts; and by $q(n, k)$ the number of partitions of n into exactly k distinct parts.

1.3. Decompositions

We have already stated that in a partition the order in which the summands appear is irrelevant. When the order of appearance of summands is relevant, the partition is called an ordered partition or a decomposition. (Some writers call it a composition.) We shall, however, invariably use the word decomposition in this sense.

The number of decompositions is usually easier to find than the number of partitions. The total number of decompositions of n is, in fact, given by 2^{n-1}. This is easily proved as follows:

The decompositions of n can be classified under two heads:

(1) Those in which the first part is 1; and

(2) Those in which the first part is > 1.

Removing 1 (the first part), from each decomposition of n of the first kind, we obtain all the different decompositions of $(n-1)$, each once. Reducing by 1 the first part in the decompositions of the second kind, we again get all the decompositions of $(n-1)$ as before. Hence, the number of decompositions of n, is twice the number of decompositions of $(n-1)$ and the result follows readily by induction.

1.4. Perfect Partitions

A partition of n is said to be perfect when it contains just one partition of every number up to n. Thus, $1+2+2+2$ is a perfect partition of 7, because every number up to 7, can be expressed uniquely as a sum by using the summands: 1, 2, 2, 2. Other such partitions of 7 are: $1+1+1+1+1+1+1$; $1+1+1+4$; $1+2+4$.

1.5. Plane and Solid Partitions

The summands in a partition of n are sometimes arranged in the form of something like a matrix, so that the elements in each row and in each column are in a descending order, though not necessarily strictly. Such an arrangement gives a plane partition of n. Thus

$$\begin{array}{cccc} 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & & \end{array}$$

is a plane partition of 18. A plane partition is also called a rowed partition. A k-rowed partition of n includes all partitions of n with k or fewer number of rows. Thus the three-rowed partitions of 4 are:

$$\begin{array}{llllllllll} 4; & 31; & 22; & 211; & 1111; & 3; & 2; & 21; & 111; & 11; \\ & & & & & 1 & 2 & 1 & 1 & 11 \end{array} \quad \begin{array}{lll} 2; & 11. \\ 1 & 1 \\ 1 & 1 \end{array}$$

The number of k-rowed partitions of n is denoted by $t_k(n)$.

In a solid partition the summands are arranged in three-dimensional space as they are arranged in a two dimensional space in the case of a plane partition. The elements are arranged in descending order of magnitude in each of the three principal directions.

2. Generating Functions

A convenient way of representing a partition of n, in which one occurs h_1 times, two h_2 times, three h_3 times and so on, is to write

$$n = 1^{h_1} 2^{h_2} 3^{h_3} \ldots n^{h_n},$$

where h's are nonnegative integers, and

$$h_1+2h_2+3h_3+\ \ldots\ +nh_n=n.$$

Now it would not be difficult to see that $p(n)$ is the coefficient of x^n in the product:

$$(1+x+x^2+\ \ldots\ +x^{h_1}+\ \ldots\ +x^n)(1+x^2+x^4+\ \ldots\ +x^{2h_2}+\ \ldots\ +x^{2[n/2]})$$

$$(1+x^3+x^6+\ \ldots\ +x^{3h_3}+\ \ldots\ +x^{3[n/3]})\ \ldots\ (1+x^n);$$

because a one-one correspondence can be set up between the partition:

$$1^{h_1}2^{h_2}3^{h_3}\ \ldots\ n^{h_n}$$

of n and the term:

$$x^{h_1}x^{2h_2}x^{3h_3}\ \ldots\ x^{nh_n}$$

of the product noted above. Using the algebra of formal power series, we can write

$$\prod_{r=1}^{\infty}(1-x^r)^{-1}=1+\sum_{n=1}^{\infty}p(n)x^n.$$

The function:

$$f(x)=\prod_{r=1}^{\infty}(1-x^r)^{-1}$$

is called the generating function for $p(n)$. In the same way, one can show that the generating function for $q(n)$ is:

$$g(x)=\prod_{r=1}^{\infty}(1+x^r).$$

In general, the generating function for the number of partitions of n into members of set

$$A=\{a_1, a_2, a_3, \ \ldots\ .\}$$

where a's are positive integers which may be considered to be arranged in a strictly ascending order of magnitude, is

$$f_A(x)=\prod_{a_j\epsilon A}(1-x^{a_j})^{-1},$$

when repetitions are allowed, and

$$g_A(x)=\prod_{a_j\epsilon A}(1+x^{a_j}),$$

when repetitions are not allowed. Moreover, if $u_A(n)$ and $v_A(n)$ denote the numbers of partitions of n into an even number of distinct a's and an odd number of distinct a's respectively, then the coefficient of x^n in the product:

$$\prod_{a_j\epsilon A}(1-x^{a_j})$$

is $u_A(n)-v_A(n)$.

The number of partitions of n into exactly k summands each taken from set A, is the coefficient of z^kx^n in

$$\prod_{a_j\epsilon A}(1-zx^{a_j})^{-1}$$

when repetitions are allowed, and in

$$\prod_{a_j \in A} (1+zx^{a_j})$$

when repetitions are not allowed. In particular, when the set A consists of all the positive integers, the generating function for $p(n, k)$ is found to be

$$x^k \prod_{r=1}^{k} (1-x^r)^{-1}.$$

Similarly the generating function for $q(n, k)$ can be shown to be

$$x^{k(k+1)/2} \prod_{r=1}^{k} (1-x^r)^{-1}.$$

If

$$1^{h_1}2^{h_2}3^{h_3} \ldots n^{h_n}$$

be a perfect partition of n, then we must have

$$(1+x+x^2+\ldots+x^{h_1})(1+x^2+x^4+\ldots+x^{2h_2})(1+x^3+\ldots+x^{3h_3}) \ldots\ldots (1+x^{nh_n})=1+x+x^2+\ldots\ldots+x^n.$$

Hence the number of perfect partitions of n, is the same as the number of solutions of the equations:

$$h_1+2h_2+3h_3+\ldots+nh_n=n, \qquad (h_1+1)(h_2+1)(h_3+1)\ldots(h_n+1)=n+1;$$

in nonnegative integers. This can be shown to be the same as the number of ordered factorizations of $(n+1)$ without unit factors. Thus for $n=7$, we have

$$n+1=8=8;\ 2.4;\ 4.2;\ 2.2.2;$$

and 7 has exactly four perfect partitions as has already been seen.

Many theorems in partitions can be easily proved by a judicious use of generating functions. For example, we have

$$\begin{aligned} g(x) &= g(x)f(x)/f(x), \\ &= \prod_{r=1}^{\infty}(1+x^r)\prod_{r=1}^{\infty}(1-x^r)^{-1} \Big/ \prod_{r=1}^{\infty}(1-x^r)^{-1}, \\ &= \prod_{r=1}^{\infty}(1-x^{2r}) \Big/ \prod_{r=1}^{\infty}(1-x^r), \\ &= \{(1-x)(1-x^3)(1-x^5)\ldots(1-x^{2r-1})\ldots\}^{-1}. \end{aligned}$$

This shows that the number of partitions of n into distinct parts is the same as the number of partitions of n into odd parts, repetitions allowed. As another example of this type of interpretation, we notice that formally we have

$$(1-x)(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^m})\ldots=1.$$

Hence

$$(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^m})\ldots=(1-x)^{-1}=1+x+x^2+x^3+\ldots+x^n+\ldots.$$

This readily shows that the number of partitions of n into distinct powers of 2 is just one, i.e., every number can be expressed as a sum of distinct powers of 2 in a unique way.

The generating function for $t_k(n)$, the number of k-rowed partitions of n, is:

$$\prod_{r=1}^{\infty}(1-x^r)^{-\min(k,r)}.$$

In 1964, Cheema and Gordon gave a combinatorial proof of this result and obtained some congruence properties of $t_2(n)$ and $t_3(n)$. Recently Gandhi has obtained some congruences for $t_k(n)$.

3. Some Important Identities

3.1. Euler's Identity

A serious study of the subject of partitions started with Euler. In 1742, Euler gave the important expansion:

$$\prod_{r=1}^{\infty}(1-x^r)=1-x-x^2+x^5+x^7-x^{12}-x^{15}+\ \ldots\ +(-1)^k\{x^{k(3k-1)/2}+x^{k(3k+1)/2}\}+\ \ldots,$$

$$=\sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k+1)/2};$$

a result which he could not, at first, prove. Using his formula he calculated a table of values of $p(n)$ for $n \leqslant 59$. I made use of this identity in extending my table of values of $p(n)$ from $n=300$ to $n=1000$. In 1830, A. M. Legendre noted that Euler's identity implies that every number which is not pentagonal, can be partitioned into an even number of distinct parts as often as it can be partitioned into an odd number of distinct integers, while the pentagonal number $k(3k+1)/2$ can be partitioned into an even number of distinct parts once oftener or once fewer times than into an odd number of distinct parts according as k is even or odd. Franklin used Ferrers graphs to give a remarkable proof of Euler's identity.

3.2. Jacobi's Identity

Jacobi gave fundamental applications of elliptic functions to the study of the theory of partitions. In 1929, he proved that

$$\prod_{r=1}^{\infty}(1-x^{2r})(1+zx^{2r-1})(1+z^{-1}x^{2r-1})=1+x(z+z^{-1})+x^4(z^2+z^{-2})+x^9(z^3+z^{-3})+\ \ldots\ .$$

He then deduced the following relations:

$$\prod_{r=1}^{\infty}(1-x^{2r})(1+x^{2r-1})^2=\sum_{-\infty}^{\infty}x^{r^2};$$

$$\prod_{r=1}^{\infty}(1-x^{2r})(1-x^{2r-1})^2=\sum_{-\infty}^{\infty}(-1)^r x^{r^2};$$

$$2x^{1/4}\prod_{r=1}^{\infty}(1-x^{2r})(1+x^{2r})^2=\sum_{-\infty}^{\infty}x^{(2r+1)^2/4}$$

and finally

$$\prod_{r=1}^{\infty}(1-x^{2r})^3=\sum_{r=0}^{\infty}(-1)^r(2r+1)x^{r^2+r}.$$

The last one of these identities is of particular importance in the theory of partitions and has been used repeatedly by Gandhi and others. Euler's identity can be shown to be a particular case of Jacobi's.

In Jacobi's identity, replace x^2 by uv, z^2 by u/v, and change u to $-u$ and v to $-v$, to obtain the identity:

$$\prod_{r=1}^{\infty}(1-u^rv^r)(1-u^rv^{r-1})(1-u^{r-1}v^r)=\sum_{-\infty}^{\infty}(-1)^r u^{r(r+1)/2}v^{r(r-1)/2}.$$

In 1964, Cheema interpreted this combinatorially in the form of the

THEOREM: *The excess of the number of partitions of* (n, m) *into even number of parts of the type* (a, a−1), (b−1, b), (c, c) *over those into odd number of parts, distinct in each case, is* $(-1)^r$ *or zero, according as* (n, m) *is or is not of the type* (r(r+1)/2, r(r−1)/2).

He also gave a graphical proof of this result.

In fact, identities involving theta functions are equivalent to combinatorial theorems involving partitions under two different conditions.

In 1966, Sudler gave two enumerative proofs of Jacobi's identity.

3.3. Cauchy's Identities

In 1843, A. Cauchy, besides proving Euler's identity, showed that

$$\prod_{r=0}^{n-1}(1+zx^r)=1+\frac{1-x^n}{1-x}z+\frac{(1-x^n)(x-x^n)}{(1-x)(1-x^2)}z^2+\ldots.$$

$$+\frac{(1-x^n)(x-x^n)(x^2-x^n)\ldots(x^{n-1}-x^n)}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^n)}z^n;$$

and

$$\prod_{r=0}^{n-1}(1-zx^r)^{-1}=1+\frac{1-x^n}{1-x}z+\frac{(1-x^n)(1-x^{n+1})}{(1-x)(1-x^2)}z^2+\frac{(1-x^n)(1-x^{n+1})(1-x^{n+2})}{(1-x)(1-x^2)(1-x^3)}z^3+\ldots\ldots$$

These identities lead to the generating functions for $p(n, k)$ and $q(n, k)$ given already. As a special case of Cauchy's identities, we have the following identities of Euler:

$$\prod_{r=0}^{\infty}(1+zx^r)\quad=1+\sum_{r=1}^{\infty}\frac{x^{r(r-1)/2}z^r}{(1-x)(1-x^2)\ldots(1-x^r)},$$

$$\prod_{r=0}^{\infty}(1-zx^r)^{-1}=1+\sum_{r=1}^{\infty}\frac{z^r}{(1-x)(1-x^2)\ldots(1-x^r)}.$$

3.4. Gordon's Identities

Basil Gordon has given the identities:

$$\prod_{r=1}^{\infty}(1-x^r)(1-yx^r)(1-y^{-1}x^{r-1})(1-y^2x^{2r-1})(1-y^{-2}x^{2r-1})=\sum_{r=-\infty}^{\infty}x^{r(3r+1)/2}(y^{3r}-y^{-3r-1}).$$

(A proof of this identity was given by Mordell in 1962.)

$$\prod_{r=1}^{\infty}(1-x^r)^3(1-x^{2r-1})^2=\sum_{r=-\infty}^{\infty}(6r+1)x^{r(3r+1)/2}.$$

(This was also proved by Bailey.)

3.5. Ramanujan's Identities

Ramanujan stated without proof the two remarkable identities:

(1) $p(4)+p(9)x+p(14)x^2+\ldots = 5\{f(x)\}^6/\{f(x^5)\}^5$;

(2) $p(5)+p(12)x+p(19)x^2+\ldots = 7\{f(x)\}^4/\{f(x^7)\}^3+49x\cdot\{f(x)\}^8/\{f(x^7)\}^7$.

These identities lead to the well known congruences of Ramanujan for the moduli $5, 5^2, 7, 7^2$; which we consider a little later. In 1950, using formal power series, D. Kruyswijk obtained the two Ramanujan identities.

3.6. Rogers-Ramanujan Identities

The following two identities, first found by Rogers in 1894, were rediscovered by Ramanujan about 1913.

(1) $$\prod_{r=0}^{\infty}\{(1-x^{5r+1})(1-x^{5r+4})\}^{-1} = 1+\prod_{r=1}^{\infty}\frac{x^{r^2}}{(1-x)(1-x^2)\ldots(1-x^r)}$$

(2) $$\prod_{r=0}^{\infty}\{(1-x^{5r+2})(1-x^{5r+3})\}^{-1} = 1+\prod_{r=1}^{\infty}\frac{x^{r(r+1)}}{(1-x)(1-x^2)\ldots(1-x^r)}.$$

Rogers-Ramanujan identities have received considerable attention. Several proofs of these identities have been given, besides two by Rogers himself. An ingenious proof was given by J. M. Dobbie in 1962. In 1954, Henry L. Alder gave a generalization of these identities. A combinatorial interpretation of the identities was given by Gordon in 1961. B. Gordon and G. E. Andrews have generalized these identities in different directions.

3.7. Generalizations of $p(n)$

In 1951, Gupta considered the function:

$$J(x)=\prod_{r=1}^{\infty}(1-x^r)^{-rj-1}.$$

This is a generalization of the generating function $f(x)$ for $p(n)$. He also considered the function:

$$J(x, m)=\prod_{r=1}^{m}(1-x^r)^{-rj-1}.$$

In 1960, O. Kolberg gave another generalization of $f(x)$ viz,

$$P(x, j)=\{f(x)\}^{-j}=1+\prod_{n=1}^{\infty}p_j(n)x^n.$$

For positive values of j, Newman has shown that $p_j(n)$ is a polynomial in j of degree n. Gupta has expressed $p_j(n)$ in terms of combinatory functions. Wright has obtained an estimate for $p_j(n)$ for large n. Gandhi has congruences for $p_j(n)$ from which results due to Ramanathan and Lahiri and also those concerning the parity of $p(n)$ follow as special cases.

Kolberg proved that

$$P(x^2, 5)P(-x, -2)=\sum_{m=-\infty}^{\infty}(3m+1)x^{m(3m+2)}$$

and

$$P(x, 5)P(x^2, -2) = \sum_{m=-\infty}^{\infty} (6m+1)x^{m(3m+1)/2}.$$

In 1953, L. Carlitz derived Newman's formula:

$$P(x, 6)P(x^5, -1) = \sum_{m=0}^{\infty} p_5(5m)x^m.$$

4. Recursion Formulas

In extending to $n=300$, the partition table which MacMahon and Ramanujan had computed up to $n=200$, I adopted a new procedure. Denoting by $r(n, m)$, the number of partitions of n with one part equal to m and other parts $\geqslant m$, I showed that

$$r(n, m) = r(n-m, m) + r(n+1, m+1).$$

This recursion formula enabled me to compute a double entry table for the values of $r(n, m)$. The work was very much reduced with the help of the following relations:

$$r(m, m) = 1;$$

$$r(m+j, m) = 0, \qquad 0 < j < m;$$

$$r(2m+j, m) = 1, \qquad 0 \leqslant j < m;$$

$$r(3m+j, m) = 2 + [j/2], \qquad 0 \leqslant j < m;$$

and

$$r(4m+j, m) = 3 + [(m+j)/2] + [j(j+6)/12], \quad 0 \leqslant j < m.$$

The result for $r(5m+j, m)$ is rather complicated. Moreover, for $0 \leqslant k < 4$,

$$r(4m+k, > m) = m + 2 + [k/2] + [(m+k+3)(m+k-3)/12].$$

Since $p(n) = r(n+1, 1)$, my table provided a table of partitions as well. The generating function for $r(n, m)$ will be readily seen to be:

$$x^m/(1-x^m)(1-x^{m+1})(1-x^{m+2}) \ldots\ldots$$

In 1958, G. Palama evaluated $r(n, > [n/5])$.
The recursion formula for $p(n, k)$ is:

$$p(n, k) = p(n-1, k-1) + p(n-k, k).$$

This gives:

$$p(n, 1) = 1;$$

$$p(n, 2) = [n/2];$$

$$p(n, 3) = [(n^2+3)/12].$$

Formulas for $p(n,k)$, $k \leq 12$, are given in the introduction to the Royal Society Tables of Partitions. In 1942, I obtained a formula expressing $r(n,m)$ in terms of $p(n,m)$. It gave

$$r(n+2m+1, m+1) = p(n,1) + p(n-m,2) + p(n-2m,3) + \ldots + p(n-jm, j+1),$$

with $j = [(n-1)/(m+1)]$.

Very recently, using elementary methods, J. Intrator has proved that the number of partitions of any natural number n into exactly k summands, is given by a polynomial of degree exactly $(k-1)$ in n, the first $[(k+1)/2]$ coefficients of which are independent of n, while the others depend on the residue of n modulo the least common multiple of the integers: 1,2,3, . . ., k. This is an improvement on earlier results. Actually the coefficient of n^j in the polynomial depends on the residue of n modulo the least common multiple of the integers: $1, 2, 3, \ldots, [k/(j+1)]$; $j \leq (k-1)$.

Let $p^*(n,k)$ denote the number of partitions of n into at most k summands. Then, we have

$$p^*(n,k) = p^*(n,k-1) + p^*(n-k,k).$$

In fact, $p^*(n,k) = p(n+k,k)$.

In 1955, Lothar Berg considering solutions of the form:

$$t^{-n} g_k(t)$$

for the above recurrence, derived many known results for $p^*(n,k)$ and for $p(n)$. In particular, he showed, in a very simple way, that

$$p(n) < C \exp(2C\sqrt{n})/\sqrt{n}, \quad C^2 = \pi^2/6.$$

5. Graphical Methods

5.1. Ferrers Graphs

An idea that has been very useful in the study of the theory of partitions, is that of Ferrers graphs. The partition: $6+2+1$ of 9, is represented by means of dots—six in the first row, two in the second and one in the third, as in the following diagram:

. . . . . .
. .
.

If we read this graph in columns, we get the partition: $3+2+1+1+1+1$ of 9. The two partitions are said to be conjugate. Some partitions are self-conjugate too. Such, for example, is the partition: $5+4+2+2+1$ of 14. Corresponding to every self-conjugate partition of n, we have a partition of n into odd distinct parts and conversely. Hence, the number of self-conjugate partitions of n is the same as the number of partitions of n into odd distinct parts. This and several other theorems in partitions are easily proved by means of graphs. For example, the following two can be mentioned:

1. The number of partitions of n into exactly k summands, is the same as the number of partitions of n into summands the largest of which is k.

2. The number of partitions of n into odd parts is the same as the number of partitions of n into distinct parts. (This has already been proved with the help of generating functions.)

5.2. Decompositions of n

Closely connected with Ferrers graphs, is the following method of representing a decomposition of n.

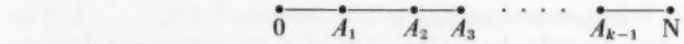
Take a straight line 0N of n units in length. If

$$n=n_1+n_2+n_3+\ \ldots\ +n_k,$$

be a decomposition of n into k parts, then taking along the line 0N;

$$0A_1=n_1,\ A_1A_2=n_2,\ A_2A_3=n_3,\ \ldots\ ,\ A_{k-1}N=n_k;$$

we get



as a graphic representation of the said decomposition. Conversely, given the graphic representation, the decomposition can be written out. It will be readily seen that the number of decompositions of n into exactly k summands, is the same as the number of ways in which exactly $(k-1)$ lattice points can be marked strictly between 0 and N. Since there are $(n-1)$ lattice points between 0 and N, in all; this can be done in $\binom{n-1}{k-1}$ ways. This gives the number of decompositions of n into k parts. The total number of decompositions of n is given by

$$\sum_{k=1}^{n}\binom{n-1}{k-1}=2^{n-1}.$$

Since a partition of n into k parts, cannot give rise to more than $k!$ decompositions (and that too only if the k parts are distinct), it follows that

$$p(n,k)\geqslant\frac{1}{k!}\binom{n-1}{k-1}\geqslant q(n,k).$$

We might mention here that the number of decompositions of n into at most k summands, is the number of solutions of the equation:

$$x_1+x_2+x_3+\ \ldots\ +x_k=n,$$

in nonnegative integers x. This is the same as the number of ways in which n like objects can be distributed among k persons when there is no restriction as to the number of objects any of them may receive.

The method of representation described above can be extended to a j-dimensional space. Gupta has thus obtained generalizations of the above inequality for partitions of j-partite numbers.

6. Congruences

6.1. Ramanujan's Conjecture

About 1919, Ramanujan proved that for any nonnegative m,

$$p(5m+4)\equiv 0(\mathrm{mod}\ 5),\qquad p(25m+24)\equiv 0(\mathrm{mod}\ 25);$$

$$p(7m+5)\equiv 0(\mathrm{mod}\ 7),\qquad p(49m+47)\equiv 0(\mathrm{mod}\ 49).$$

He further stated that

$$p(11m+6)\equiv 0(\mathrm{mod}\ 11).$$

All these results are included in his famous conjecture:

$$\text{If } p=5,\ 7,\ \text{or } 11 \text{ and } (24n-1)\equiv 0 \pmod{p^a},\ a\geq 1;$$

then also $p(n)\equiv 0 \pmod{p^a}$.

This astounding conjecture held good till, using my table of values of $p(n)$ for $n\leq 300$, S. Chowla found that the conjecture failed for $n=243$. For this value of n,

$$24n-1=5831\equiv 0 \pmod{7^3},$$

while

$$p(243)=13397\quad 82593\quad 44888\equiv 0 \pmod{7^2},$$

but

$$\not\equiv 0 \pmod{7^3}.$$

Later it was found that the conjecture failed also when $n=586$.

This discovery led to a lot of work particularly by D. H. Lehmer, who using Hardy-Ramanujan and Rademacher Series (discussed later in this survey), computed values of $p(n)$ for certain large values of n. He thus found that

$$p(599)\equiv 0 \pmod{5^4},\qquad p(721)\equiv 0 \pmod{11^3};$$
$$p(1224)\equiv 0 \pmod{5^4},\qquad p(2052)\equiv 0 \pmod{11^3};$$
$$p(2474)\equiv 0 \pmod{5^5},\qquad p(14031)\equiv 0 \pmod{11^4}.$$

These results are in conformity with Ramanujan's conjecture.

In 1948, D. B. Lahiri found three new congruences:

$$p(49m+r)\equiv 0 \pmod{49} \text{ for } r=19,\ 33,\ 40;$$

and announced two more:

$$p(125m+r)\equiv 0 \pmod{125} \text{ for } r=74,\ 124.$$

In 1952, from a study of Ramanujan's manuscript, J. M. Rushforth proved that

$$p(121m+116)\equiv 0 \pmod{121};$$

and

$$p(49m+r)\equiv 0 \pmod{49} \text{ for } r=19,\ 33,\ 40,\ 47.$$

In 1960, Professor M. Newman showed that for $(m,\ 30)=1$.

$$p((167m^2+1)/24)\equiv 0 \pmod{5}.$$

These were all results of the Ramanujan type.

In the meantime, G. N. Watson had proved Ramanujan's conjecture completely for powers of 5. For powers of 7, he proved the following modification of the conjecture:

If $$24n-1\equiv 0 \pmod{7^b},$$

then $$p(n)\equiv 0 \pmod{7^d} \text{ where } d=[(b+2)/2].$$

Finally, A. O. L. Atkin has essentially settled the problem by proving Ramanujan's conjecture for powers of 11. The full truth with regard to the conjecture can now be stated in the form of the

THEOREM: *If* $24n-1 \equiv 0 \pmod{5^a 7^b 11^c}$, *where* a, b, c *are nonnegative integers, then*

$$p(n) \equiv 0 \pmod{5^a 7^d 11^c} \text{ with } d = [(b+2)/2].$$

Watson's method was based on modular equations. Atkin followed the method of J. Lehner, which had enabled Lehner to prove Ramanujan's conjecture for 11, 11^2, 11^3, in 1943, 1950.

Mention must also be made of the method which Professor H. Rademacher developed in 1942, for the investigation of the Ramanujan identities with respect to the moduli 5^a, 7, 7^2, and 13^a.

Very recently, A. O. L. Atkin and J. N. O'Brien have made a valuable contribution to the literature on the subject by giving some properties of $p(n)$ modulo powers of 13.

6.2. The Parity of $p(n)$

In 1920, MacMahon had shown that

$$p(n) \equiv \sum_t p(t) \pmod 2$$

where t runs through the positive integral values given by the relation:

$$8t = 2n - j(j+1), \qquad j \geq 0.$$

It is rather strange that nothing better should be known about the parity of $p(n)$ than MacMahon's congruence.

In 1946, making use of the fact that Ramanujan's function $\tau(n)$ is odd if and only if n is an odd square, I obtained MacMahon's congruence.

6.3. Congruences for $p_j(n)$

Newman, Kolberg, Ramanathan, Gandhi and several others have studied the function $p_j(n)$ defined in section 3.7. In 1956, Newman gave extensive tables of values of $p_j(n)$ for $j \leq 16$.

For $j = 2, 4, 6, 8, 10, 14, 26$ and a prime $p > 3$ such that

$$j(p+1) \equiv 0 \pmod{24}.$$

M. Newman proved that

$$p_j(np+k) = (-p)^{(j-2)/2} p_j(n/p),$$

where $k = j(p^2-1)/24$ and we take $p_j(\alpha) = 0$ if α is not a nonnegative integer. He also showed that

$$p_6(3n+2) = 9p_6(n/3);$$

and that $p_j(n)$ vanishes for arbitrarily long strings of consecutive values of n, when $j = 2, 4, 6, 8, 10, 14$ or 26.

Ramanathan showed that

If $24n + j \equiv 0 \pmod{5^a}$, $j \equiv 16, 21, 26 \pmod{30}$.

then $$p_j(n) \equiv 0 \pmod{5^b} \text{ where } b = [(a+1)/2].$$

6.4. Rank of a Partition

In 1954, Atkin and Swinnerton-Dyer proved some conjectures of Dyson, who had discovered, in 1944, empirically a remarkable combinatorial method of dividing the partitions of $(5n+4)$ and of $(7n+5)$ into five and seven equal classes respectively. Dyson defined the rank of a partition as the largest part minus the number of parts in the partition. The partitions of a given number could then be divided into classes according to their ranks. In the case of numbers of the forms $(5n+4)$ and $(7n+5)$, these classes had equal number of elements in each class.

Atkin and Swinnerton-Dyer showed that for $q=5$, 7, or 11, and $0<b<q$;

$$\sum_{n=0}^{\infty} p(qn+b)y^n$$

is congruent modulo q to a simple infinite product.

In 1958, Atkin and Hussain obtained an identity for

$$\sum_{n=0}^{\infty} p(11n+6)y^n.$$

6.5. Sylvester's Theorem

In 1945, D. P. Banerjee proved Sylvester's theorem, which states that $p(n)-Q(n)$ is even, where $Q(n)$ denotes the number of partitions of n into odd distinct parts.

6.6. Newman's Conjecture

In 1960, M. Newman conjectured that for every pair of natural numbers m and r, there are infinitely many natural numbers n, such that

$$p(n) \equiv r(\text{mod } m).$$

He proved the conjecture for $m=5$, 13, 65 and some other composite moduli. Kolberg and Subbarao have proved it for $m=2$. In this connection, the following two results of Newman might be of interest:

Neither of the congruences:

$$p(13n-7) \equiv r(\text{mod } 13),$$

and

$$p(13^2n-7) \equiv r(\text{mod } 13),$$

can hold for all sufficiently large n and a fixed r.

7. Some Inequalities

We have already shown that

$$k!q(n,k) \leqslant \binom{n-1}{k-1} \leqslant k!p(n,k).$$

Since,

$$q(n,k) = p(n-\tfrac{1}{2}k(k-1),k),$$

we must have

$$\binom{n+\frac{1}{2}k(k-1)-1}{k-1} \geqslant k!p(n,k) \geqslant \binom{n-1}{k-1}.$$

(Originally, I had made use of the recursion formula for $p(n,k)$ to prove this inequality.)

The inequality implies that

$$p(n, k) \sim \frac{1}{k!}\binom{n-1}{k-1}, \text{ for } k=o(n^{1/3}).$$

Erdos and Lehner had proved this result in 1941, by a rather lengthy method. They had in fact proved that if $p^*(n, k)$ denotes the number of partitions of n into at most k summands, then

$$p^*(n, k)/p(n) \sim \exp(-C^{-1}e^{-Cx}),$$

where (here and in all that follows), $C=\pi/\sqrt{6}$; and x is given by the relation:

$$Ck=\sqrt{n}\log\sqrt{n}+Cx\sqrt{n}.$$

They gave a similar result for partitions into distinct parts.

As an analogue of this remarkable result of Erdos and Lehner, I showed that

$$p(n, k)/p(n) \sim tn^{-1/2}\exp(-t/C).$$

where

$$t=\lim_{n,k\to\infty}\sqrt{n}\exp(-Ckn^{-1/2}).$$

In 1957, S. M. Luthra obtained an asymptotic formula for

$$\sum_{k=1}^{n} kp(n, k)/p(n).$$

The case of partitions of n into k distinct parts was also considered.

In 1950, Auluck, Singwi, and Agarwala gave another asymptotic result of some interest:

If $w(n)$ denotes the number of partitions of n into integers each of which occurs in the partition only an odd number of times, then for a certain constant γ,

$$w(n) \sim \frac{\gamma}{2\pi n}\exp(2\gamma\sqrt{n}).$$

If $p(n, A)$ denotes the number of partitions of n into members of the set:

$$A=\{1, a_1, a_2, a_3, \ldots, a_m\},$$

then, I proved in 1955, the inequality:

$$\binom{(n+m)}{m} \leqslant p(n, A)\prod_{j=1}^{m} a_j \leqslant \binom{n+\sum_{j=1}^{m} a_j}{m}.$$

This is a generalization of the inequality given earlier.

In 1954, writing

$$\{(1-x)(1-x^2)(1-x^3)\ldots(1-x^r)\}^{-1}=\sum_{j=0}^{\infty} c_j(r)(1-x)^{j-r},$$

I showed that for a fixed j and large r,

$$c_j(r) \sim \{j!\,(r-2j)!\,4^j\}^{-1}.$$

This led to an asymptotic formula for $p^*(n, r)$.

In 1956, Bateman and Erdos showed that the number of partitions of n into parts taken from a nonempty set A of positive integers, is strictly increasing for large n, if and only if A contains more than one element and if when any element is removed from A, the remaining elements have the greatest common divisor unity.

8. Conjecture of Auluck, Chowla, and Gupta

In 1942, Auluck, Chowla, and Gupta conjectured that $p(n, k)$ has a unique maximum for any given n, i.e., regarding n as fixed there exists an integer k_0 such that

$$p(n, k) \geqslant p(n, k-1), \text{ for } k \leqslant k_0;$$

and

$$\leqslant p(n, k-1), \text{ for } k > k_0.$$

This conjecture has not been completely proved so far.

In 1953, however, G. Szekeres proved that a maximum of $p(n, k)$ occurs at

$$k=\frac{\sqrt{n}}{C}L+\frac{1}{C^2}\left(\frac{3}{2}+\frac{3}{2}L-\frac{1}{4}L^2\right)-\frac{1}{2}+\mathbf{O}\left(\frac{\log^4 n}{\sqrt{n}}\right),$$

where

$$L=\log\,(\sqrt{n}/C).$$

He deduced this result from his formula for $p^*(n, k)$ true for bounded (n/k^2). The formula itself was obtained by using the method of steepest descent.

In fact, the work of Szekeres opened up the possibility of obtaining the infinite series for $p(n)$ without using the theory of elliptic modular functions. He sharpened the results of Erdos and Lehner regarding maximum of $p(n, k)$ and $q(n, k)$. It was shown that $q(n, k)$ has a maximum at

$$k=\frac{2\sqrt{3n}\,\log 2}{\pi}+\beta+\mathbf{O}(n^{-1/2});$$

where β is a fixed constant.

In 1954, Haselgrove and Temperley developed a powerful method which made no use of Farey dissections and enabled them to prove that $p(n, k)$ attains its greatest value for at most two consecutive values of k when n is large and fixed. This had been conjectured by Auluck, Chowla, and Gupta.

9. Hardy-Ramanujan and Rademacher Series

No account of partition theory can be even fairly complete without a mention of the great work of Hardy and Ramanujan on the subject. Using Farey dissections, they obtained an infinite series for $p(n)$, the first few terms of which gave the value of $p(n)$ exactly if we neglect the decimal part in the answer. This was astounding. Later on Lehmer showed that the Hardy-Ramanujan series was divergent and Hardy and Ramanujan had been fortunate in breaking at a point where the series gave a correct answer.

The Hardy-Ramanujan series gives

$$p(n)=\frac{(12)^{1/2}}{(24n-1)u}\sum_{k=1}^{[\sqrt{n}]}A_k^*(n)\,(u-k)\exp\,(u/k)+\mathbf{O}(\log n/\sqrt{n});$$

where

$$u=\pi(24n-1)^{1/2}/6.$$

A change in the path of integration enabled Rademacher to replace $(u-k)\exp(u/k)$ in the Hardy-Ramanujan series by

$$(u-k)\exp(u/k)+(u+k)\exp(-u/k).$$

This simple-looking change gave a convergent series for $p(n)$.

The functions $A_k^*(n)$ in the series for $p(n)$ have been shown by Lehmer to be multiplicative in a certain sense. In 1956, Whiteman used Fourier series for the evaluation of these functions and Rademacher obtained the Selberg formula for $A_k^*(n)$ by using a transformation. He also put Lehmer's theorems for the evaluation and factorization of $A_k^*(n)$ in a form suitable for computation.

The method of Rademacher has been widely applied to partition problems of various types. For example, Lehner has obtained a convergent series for the number of partitions of a positive integer n into summands of the form: $5m\pm a$, $a=1, 2$; and Hua for $q(n)$. Hagis has not only derived Rademacher's formula for $p(n)$ and the convergent series for $q(n)$, but also a convergent series for $q(n; t)$—the number of partitions of n into odd summands, no summand appearing more than t times in a partition.

In 1941, Ingham used a Tauberian theorem to obtain the asymptotic formulas for $p(n)$ and $q(n)$. These are:

$$p(n)\sim(4n\sqrt{3})^{-1}\exp(\pi\sqrt{2n/3}),$$
$$q(n)\sim(4n^{3/2}3^{1/4})^{-1}\exp(\pi\sqrt{n/3}).$$

In 1961, Iseki gave a shorter proof than Rademacher's for a functional equation which implies the transformation equation for Dedekind's modular function $\eta(\tau)$. D. J. Newman has obtained the asymptotic formula for $p(n)$ in an elementary and simple way. In 1962, he gave another proof for it.

In 1954, Petersson developed a function-theoretic method for finding an asymptotic formula for the partition function. This method is different from the Hardy-Ramanujan method.

10. Miscellaneous Results

Mention must be made of the work of O. P. Gupta and S. Luthra who, in 1955, gave a table for the number of partitions of $n\leq 300$, into primes. I gave a table for the number of partitions into distinct primes.

In 1957, Takayoshi Mitsui used the powerful methods of Vinogradov and Hua to obtain an asymptotic formula for the number of partitions of n into kth powers of primes not exceeding m. His results generalized those of Haselgrove and Temperley.

Before I conclude, I give some results which, I believe, will interest the reader.

(i) The number of partitions of n into distinct parts, the smallest being odd, is odd if and only if n is a square. This was given by N. J. Fine in 1948.

(ii) In 1958, Guy proved that the number of partitions of an integer into:

(a) odd parts greater than unity;

(b) unequal parts such that the two greatest parts differ by unity;

(c) unequal parts which are not powers of 2; are all equal.

(iii) In 1963, R. L. Graham showed that every integer greater than 77 can be partitioned into distinct positive integers whose reciprocals add to 1.

(iv) J. B. Kelly proved the following result in 1964: Given $k\geq 3$, every $n\geq N(k)$, can be partitioned into k parts in $(k-1)$ different ways so that the products of the integers in each of the $(k-1)$ partitions are equal.

(v) By far the most remarkable result regarding two-rowed partitions of n, when the elements in rows strictly decrease, is that the number of such partitions is $p(n)$, but it is only $p([n/2])$ when the parts are all odd.

This was proved by B. Gordon in 1962. Sudler gave a direct proof of this result in 1965. More recently B. Gordon and L. Houten obtained generating functions and asymptotic formulas for various restricted rowed and plane partitions.

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A Relation Between the Coefficients and Roots of Two Equations and Its Application to Diophantine Problems*

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The object of this paper is to prove:
THEOREM 1: *The system of equations*

$$a_1^r + a_2^r + \ldots + a_n^r = b_1^r + b_2^r + \ldots + b_n^r (r = 1, 2, \ldots, j-1, j+1, \ldots, n+1)$$

where j *is odd, has no nontrivial solutions in positive integers.*
THEOREM 2: *No two equations* $a_1^r + a_2^r = b_1^r + b_2^r (r = s, t)$ *where the pairs of values of* r *range between* 1 *and* 4 *can have a nontrivial solution in integers.*

Key words: Diophantine equations; Prouhet-Tarry-Escott problem; symmetric functions.

In section 1 of this paper we indicate a general relation between the coefficients of two polynomial equations of the same degree $n \geq 1$ and the differences of the sums of like powers of the roots of the equations. In section 2 we apply this relation and deduce some results concerning the problem of equal sums of like powers in number theory.

1. Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be two sets of numbers, not all necessarily rational in any set. Let the elements a_i of the first set be the roots of the equation

$$x^n - p_1 x^{n-1} + \ldots + (-1)^n p_n = 0 \tag{1}$$

and so also let the elements b_i of the second set be the roots of the equation

$$y^n - q_1 y^{n-1} + \ldots + (-1)^n q_n = 0. \tag{2}$$

In what follows we write

$$A_r = a_1^r + a_2^r + \ldots + a_n^r, \; B_r = b_1^r + b_2^r + \ldots + b_n^r \tag{3}$$

where r is an integer ≥ 1. By Newton's formula on the sum of the powers of the roots of an equation we have from equation (1),

$$A_r - p_1 A_{r-1} + p_2 A_{r-2} - \ldots + (-1)^{r-1} p_{r-1} A_1 + (-1)^r r p_r = 0 \tag{4}$$

for $r = 1, 2, 3, \ldots$ with $p_r = 0$ if $r > n$. Similarly, from equation (2)

$$B_r - q_1 B_{r-1} + q_2 B_{r-2} - \ldots + (-1)^{r-1} q_{r-1} B_1 + (-1)^r r q_r = 0 \tag{5}$$

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for $r=1, 2, 3, \ldots$ with $q_r=0$ if $r>n$. Eliminating $p_1, p_2, \ldots p_{r-1}$ between the first r equations in (4), we easily obtain

$$r!p_r=\begin{vmatrix} A_1 & 1 & 0 & 0 & 0 \ldots 0 \\ A_2 & A_1 & 2 & 0 & 0 \ldots 0 \\ A_3 & A_2 & A_1 & 3 & 0 \ldots 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ A_{r-1} & A_{r-2} & A_{r-3} & A_{r-4} & A_{r-5} \ldots (r-1) \\ A_r & A_{r-1} & A_{r-2} & A_{r-3} & A_{4-4} \ldots A_1 \end{vmatrix} \tag{6}$$

with a similar expression for $r!q_r$. Let us write

$$A_r=B_r+h_r \qquad (r=1, 2, 3, \ldots) \tag{7}$$

and let H_r denote the determinant obtained by replacing A_j by h_j in the determinant in (6). When the A_j are so replaced, $r!p_r$ is replaced by H_r and by symmetry we have, corresponding to the relation (4),

$$h_r-\frac{H_1}{1!}h_{r-1}+\frac{H_2}{2!}h_{r-2}-\ldots\ldots+(-1)^r r\frac{H_r}{r!}=0. \tag{8}$$

Writing $c_r=r!p_r$ and $d_r=r!q_r$ we then get

$$c_1=d_1+H_1,\ H_1=h_1$$

$$c_2=A_1^2-A_2=(B_1+h_1)^2-(B_2+h_2)=(B_1^2-B_2)+2h_1B_1+h_1^2-h_2$$

$$=d_2+2H_1d_1+H_2=d_2+\binom{2}{1}H_1d_1+\binom{2}{2}H_2$$

$$c_3=A_1^3-3A_1A_2+2A_3=(B_1+h_1)^3-3(B_1+h_1)(B_2+h_2)+2(B_3+h_3)$$

$$=d_3+3H_1d_2+3H_2d_1+H_3=d_3+\binom{3}{1}H_1d_2+\binom{3}{2}H_2d_1+\binom{3}{3}H_3.$$

Let us assume that

$$c_m=d_m+\binom{m}{1}H_1d_{m-1}+\binom{m}{2}H_2d_{m-2}+\ldots\ldots+\binom{m}{m}H_m$$

for $m=1, 2, \ldots., (r-1)$, $r\geq 2$. Then (4) gives by virtue of (7),

$$(B_r+h_r)-\frac{1}{1!}(d_1+H_1)(B_{r-1}+h_{r-1})+\frac{1}{2!}\left[d_2+\binom{2}{1}H_1d_1+\binom{2}{2}H_2\right](B_{r-2}+h_{r-2})-\ldots\ldots$$

$$\ldots+(-1)^{r-1}\frac{1}{(r-1)!}\left[d_{r-1}+\binom{r-1}{1}H_1d_{r-2}+\ldots+\binom{r-1}{r-1}H_{r-1}\right](B_1+h_1)$$

$$+(-1)^r\frac{rc_r}{r!}=0,\ r\geq 2. \tag{9}$$

If we denote the left-hand side of (5) and (8) by Q_r and T_r respectively, then arranging the terms in (9) we get

$$\left[Q_r-(-1)^r\frac{rd_r}{r!}\right]-\frac{H_1}{1!}\left[Q_{r-1}-(-1)^{r-1}\frac{(r-1)d_{r-1}}{(r-1)!}\right]+\ldots$$

$$+(-1)^{r-1}\frac{H_{r-1}}{(r-1)!}\left[Q_1-(-1)^1\frac{1d_1}{1!}\right]+\left[T_r-(-1)^r\frac{rH_r}{r!}\right]-\frac{d_1}{1!}\left[T_{r-1}-(-1)^{r-1}\frac{(r-1)H_{r-1}}{(r-1)!}\right]$$

$$+\ldots+(-1)^{r-1}\frac{d_{r-1}}{(r-1)!}\left[T_1-(-1)^1\frac{1H_1}{1!}\right]+(-1)^r\frac{rc_r}{r!}=0.$$

Noting that $Q_j=T_j=0$, $j=1, 2, 3, \ldots$, the above easily gives

$$-\frac{rd_r}{r!}-\frac{H_1}{(r-1)!1!}(r-1+1)d_{r-1}-\frac{H_2}{(r-2)!2!}(r-2+2)d_{r-2}-\ldots.$$

$$-\frac{H_{r-1}}{1!(r-1)!}d_1-\frac{rH_r}{r!}+\frac{rc_r}{r!}=0$$

that is,

$$c_r=d_r+\binom{r}{1}H_1d_{r-1}+\binom{r}{2}H_2d_{r-2}+\ldots\ldots+\binom{r}{r}H_r \tag{10}$$

Hence, by induction, it follows that (10) holds for all $r=1, 2, 3, \ldots$, (where, of course, $c_r=d_r=0$ if $r>n$).

2. We now restrict the elements a_i and b_i of the preceding section to be all integers. The notations continue to remain the same as before and in what follows now all letters denote integers. If we put in (7) $h_r=0$ for $r=s, t, \ldots$, there results a system of equations with equal sums of like powers of the type

$$A_r=B_r(r=s, t, \ldots) \tag{11}$$

where A_r and B_r are given by (3). Conversely, a system (11) makes in (7) $h_r=0$, $r=s, t, \ldots$. Using the relation (10) of the previous section we shall now deduce that certain systems (11) do not possess nontrivial solutions in integers.

I. If the a's and b's satisfy (11) with $r=1, 2, \ldots, n$, then since $h_j=0(j=1, 2, \ldots n)$, the relation (10) at once yields the well known Bastien's theorem [1][1]

II. Let the a_i and b_i be positive integers which satisfy the system (11) for $r=1, 2, \ldots, j-1$, $j+1, \ldots, n+1$. Then c_r and d_r are all positive for $r=1, 2, \ldots, n$ and we have $h_1=h_2=\ldots\ldots=h_{j-1}=h_{j+1}\ldots\ldots=h_{n+1}=0$. Without loss of generality we may suppose $h_j\geq 0$. Then by (8)

$$H_r=(-1)^{j-1}\frac{(r-1)!}{(r-j)!}H_{r-j}h_j \tag{12}$$

From the relation (12), it follows that $H_r=0$ for $r=1, 2, \ldots, (j-1)$ and for $r=j$

$$H_j=(-1)^{j-1}(j-1)!h_j \tag{13}$$

If $r>j$, the same relation shows that (since H_r is expressed in terms of H_{r-j} and $H_r=0$ for $r=1, 2, \ldots (j-1))H_r=0$ unless r is a multiple of j. However, if r is a multiple of j, say $r=jk$, we have by repeated application of (12)

$$H_{jk}=(-1)^{(j-1)(k-1)}\frac{(jk-1)!(jk-j-1)!\ldots(2j-1)!}{(jk-j)!(jk-2j)!\ldots j!}H_jh_j^{k-1}$$

$$=(-1)^{(j-1)k}\frac{(jk-1)!(jk-j-1)!\ldots(2j-1)!}{(jk-j)!(jk-2j)!\ldots j!}(j-1)!h_j^k, \tag{13} \tag{14}$$

Now if j is odd, we have by (14), $H_{jk}\geq 0$, the sign of equality occurring only when $h_j=0$. Assuming that j is odd, if we substitute the values of $H_r(r=1, 2, \ldots, n+1)$, (which is either zero or positive as obtained in (14)), in the relation (10) with $r=n+1$, then, since $c_{n+1}=d_{n+1}=0$, the L.H.S. is zero while the R.H.S. would be positive unless $h_j=0$. In the latter case, however, the solutions are only trivial by Bastien's theorem. We have thus proved

THEOREM 1: *The system of equations*

$$a_1^r+a_2^r+\ldots+a_n^r=b_1^r+b_2^r+\ldots+b_n^r(r=1, 2, \ldots, j-1, j+1, \ldots, n+1),$$

where j *is odd, has no nontrivial solutions in positive integers.*

[1] Figures in brackets indicate the literature references at the end of this paper.

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This confirms an assertion of Gupta (sec. 3, [3]) made there without proof. The case $j=1$ gives precisely his theorem (sec. 1 [3]).

III. To treat the pair of equations

$$a_1^r+a_2^r=b_1^r+b_2^r \qquad (r=s,\ t) \tag{14}$$

we may apply the relations of the previous section with $n=2$. Without loss of generality one can suppose that s and t are relatively prime in (15). For sake of simplicity we consider here only such pairs of values of r in (15) as range between 1 and 4. We then need consider the system (15) only with the following pairs of values of r:

$$r=1,2; \qquad 1,3; \qquad 1,4; \qquad 3,4; \qquad 2,3.$$

(a) $r=1,\ 2$. In this case, the system (15) has only trivial solutions by Bastien's theorem.

(b) $r=1,3$. In this case, the system (15) implies the system

$$(w+a_1)^r+(w+a_2)^r+(w-b_1)^r+(w-b_2)^r=(w+b_1)^r+(w+b_2)^r+(w-a_1)^r+(w-a_2)^r$$

for $r=1,\ 2,\ 3$ and 4, where w is arbitrary. The latter system has only trivial solutions by Bastien's theorem and so has the system (15) in this case.

In each of the three remaining cases we actually calculate the coefficients of the quadratic equations (1) and (2) in terms of the h's by means of the relation (10) and then apply the condition for integral roots: $c_1^2-2c_2=\square, \qquad d_1^2-2d_2=\square$

(c) $r=1,\ 4$. In this case $h_1=h_4=0$ and (10) gives

$$c_1=d_1, \qquad c_2=d_2-h_2$$

$$0=-3h_2d_1+2h_3, \qquad 0=-6h_2d_2+8h_3d_1+3h_2^2 \tag{16}$$

We may suppose $h_2\neq 0$, for otherwise (15) has only trivial solutions. From (16) we then obtain

$$c_1=d_1=2h_3/3h_2,\ c_2=(16h_3^2-9h_2^3)/18h_2^2,\ d_2=(16h_3^2+9h_2^3)/18h_2^2.$$

Applying the condition for integral roots, we have

$$-12h_3^2+9h_2^3=u_1^2, \qquad -12h_3^2-9h_2^3=u_2^2. \tag{17}$$

By addition, we see that (17) can hold only with $h_3=0$, in which case (15) has only trivial solutions by case (b). The same conclusion was arrived at by Gloden [2], using a different method.

(d) $r=3,\ 4$. In this case $h_3=h_4=0$ and (10) gives

$$c_1=d_1+h_1, \qquad c_2=d_2+2h_1d_1+h_1^2-h_2, \qquad 0=3h_1d_2+3(h_1^2-h_2)d_1+h_1^3-3h_1h_2$$

$$0=6(h_1^2-h_2)d_2+4(h_1^3-3h_1h_2)d_1+h_1^4-6h_1^2h_2+3h_2^2. \tag{18}$$

We may suppose that $h_1\neq 0$, for otherwise (15) has only trivial solutions by case (b). From (18) we then obtain

$$c_1=h_1(h_1^4+2h_1^2h_2+3h_2^2)/2(h_1^4+3h_2^2),$$

$$d_1=h_1(-h_1^4+2h_1^2h_2-3h_2^2)/2(h_1^4+3h_2^2)$$

$$c_2=(h_1^6+3h_1^4h_2+9h_1^2h_2^2-9h_2^3)/6(h_1^4+3h_2^2),$$

$$d_2=(h_1^6-3h_1^4h_2+9h_1^2h_2^2+9h_2^3)/6(h_1^4+3h_2^2).$$

Applying the condition for integral roots, we have

$$3h_1^2(h_1^4 \pm 2h_1^2h_2 + 3h_2^2)^2 - 4(h_1^4 + 3h_2^2)(h_1^6 \pm 3h_1^4h_2 + 9h_1^2h_2^2 \mp 9h_2^3) = 12v_1^2,\ 12v_2^2.$$

Adding these two equations and simplifying, we get

$$-h_1^2(h_1^4 + 9h_2^2)^2 = 6(v_1^2 + v_2^2)$$

which can hold only with $h_1 = 0$, contrary to assumption. It follows that (15) has only trivial solutions in this case.

(e) $r = 2, 3$. In this case $h_2 = h_3 = 0$ and (10) gives

$$c_1 = d_1 + h_1, \qquad c_2 = d_2 + 2h_1d_1 + h_1^2, \qquad 0 = 3h_1d_2 + 3h_1^2d_1 + h_1^3 = 6h_1^2d_2 + 4h_1^3d_1 + h_1^4 - 6h_4. \tag{19}$$

We may assume $h_1 \neq 0$, for otherwise we have only trivial solutions. Then (19) gives

$$c_1 = (h_1^4 - 6h_4)/2h_1^3, \qquad d_1 = -(h_1^4 + 6h_4)/2h_1^3, \qquad c_2,\ d_2 = (h_1^4 \mp 18h_4)/6h_1^2. \tag{20}$$

Applying the condition for integral roots

$$-h_1^8 \pm 36h_1^4h_4 + 108h_4^2 = 12w_1^2,\ 12w_2^2. \tag{21}$$

Dividing by 3 and putting $6h_4 = f$, $h_1^4 = g$ where h_1 and hence g is divisible by 3, $2w_1 = u$, $2w_2 = v$ we have from (21)

$$u^4 + u^2v^2 + v^4 = 3z^2, \qquad \text{where } z = f^2 - g^2/3. \tag{22}$$

The equation (22) is known to be impossible in integers except when $u^2 = v^2 = z$ (see for instance [5]). In the latter case, however, $w_1^2 = w_2^2$ and by substraction (21) gives $h_1h_4 = 0$. Thus, as $h_1 \neq 0$ by assumption, $h_4 = 0$, in which case (15) has only trivial solutions by case (d). The same result was obtained by Schmidt [4] as a byproduct in discussing the rational points on curves of genus 1. The preceding conclusions are summarized in:

THEOREM 2: *No two equations (15) where the pairs of values of* r *range between 1 and 4 can have a nontrivial solution in integers.*

Addendum: Theorem 1 is also true when j is even $(1 < j \leqslant n+1)$.

PROOF. Let j be even $(1 < j \leqslant n+1)$. We may assume $h_j > 0$ (the case $h_j = 0$ gives only trivial solutions). If we set $n+1 = qj + R$, $0 \leqslant R < j$, then on putting $r = n+1$ in (10) we have (since $H_r = 0$ unless $r = kj$)

$$0 = \sum_{k=1}^{q} \binom{n+1}{kj} H_{kj}d_{n+1-kj} \qquad \text{with} \qquad d_{n+1-qj} = d_0 = 1 \qquad \text{if} \qquad n+1 = qj.$$

If $q = 1$, this equation is impossible, since the d's are positive and with j even. $H_j < 0$. Hence we take $q > 1$. Substituting for H_{kj} $(k = 1, 2, \ldots, q)$ from (14) and dividing by $(j-1)!h_j$ we get from the above equation

$$\binom{n+1}{j} d_{n+1-j} = \sum_{k=2}^{q} (-1)^k \binom{n+1}{kj} \frac{(kj-1)!(kj-j-1)! \ldots (2j-1)!}{(kj-j)!(kj-2j)! \ldots j!} h_j^{k-1}d_{n+1-kj}$$

$$= \sum_{k=2}^{q} (-1)^k \frac{(n+1)!}{(n+1-kj)!} \cdot \frac{h_j^{k-1}d_{n+1-kj}}{k!j^{k-1} \cdot j!}$$

whence

$$d_{n+1-j} = (n+1-j)! \sum_{k=2}^{q} (-1)^k \frac{h_j^{k-1}d_{n+1-kj}}{(n+1-kj)!k!j^{k-1}} \tag{14$_1$}$$

Again from (10) with $r=n+1-j$ we have

$$c_{n+1-j}=d_{n+1-j}+\sum_{k=2}^{q}\binom{n+1-j}{k-1j}H_{\overline{k-1}j}d_{n+1-kj}.$$

Substituting for $H_{\overline{k-1}j}$ and simplifying as above we have, since the c's are positive,

$$d_{n+1-j}>\sum_{k=2}^{q}(-1)^k\frac{(n+1-j)\,!}{(n+1-kj)\,!}\cdot\frac{h_j^{k-1}d_{n+1-kj}}{(k-1)\,!\,j^{k-1}} \tag{14_2}$$

Hence from (14_1) and (14_2) we have

$$\sum_{k=2}^{q}(-1)^k\frac{h_j^{k-1}d_{n+1-kj}}{(n+1-kj)\,!k!j^{k-1}}>\sum_{k=2}^{q}(-1)^k\frac{h_j^{k-1}d_{n+1-kj}}{(n+1-kj)\,!(k-1)\,!j^{k-1}}.$$

Transposing and removing the factor h_j/j we have from above

$$\sum_{k=2}^{q}(-1)^{k-1}\frac{(k-1)h_j^{k-2}d_{n+1-kj}}{(n+1-kj)\,!\,k\,!\,j^{k-2}}>0 \tag{14_3}$$

whence

$$-d_{n+1-2j}+2!(n+1-2j)\,!\sum_{k=3}^{q}(-1)^{k-1}\frac{(k-1)h_j^{k-2}d_{n+1-kj}}{(n+1-kj)\,!k!j^{k-2}}>0 \tag{14_4}$$

Again from (10) with $r=n+1-2j$ we have

$$c_{n+1-2j}=d_{n+1-2j}+\sum_{k=3}^{q}\binom{n+1-2j}{k-2j}H_{\overline{k-2}j}d_{n+1-kj}$$

and therefore as above

$$d_{n+1-2j}+(n+1-2j)\,!\sum_{k=3}^{q}(-1)^{k-2}\frac{k(k-1)h_j^{k-2}d_{n+1-kj}}{(n+1-kj)\,!\,k\,!\,j^{k-2}}>0. \tag{14_5}$$

Adding (14_4) and (14_5) and removing the factor $(n+1-2j)\,!h_j/j$ we get

$$\sum_{k=3}^{q}(-1)^{k-2}\frac{(k-1)(k-2)h_j^{k-3}d_{n+1-kj}}{(n+1-kj)\,!k!j^{k-3}}>0. \tag{14_6}$$

Repeating the process, which reduces (14_3) to (14_6), with (14_6) and so in succession with each such resulting inequality we ultimately get

$$(-1)^{q-(q-1)}\cdot\frac{(q-1)(q-2)\ \ldots\ (q-\overline{q-1})\,h_j^{q-q}d_{n+1-qj}}{(n+1-qj)\,!q!j^{q-q}}>0$$

i.e., $-d_{n+1-qj}>0$, which is clearly impossible since $d_{n+1-qj}\geq 1$.

I thank Professor H. Gupta who kindly gave some useful suggestions in the preparation of this paper.

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(Paper 74B1–314)

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An Enumeration Problem for a Congruence Equation*

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(September 9, 1969)

It is shown that the number of n-tuples $(x_0, x_1, \ldots, x_{n-1})$ of nonnegative integers such that

$$\sum_{i=0}^{n-1} x_i = n,$$

$$\sum_{i=0}^{n-1} i x_i \equiv 0 \bmod n,$$

is given by

$$\frac{1}{n}\sum_{d|n} \binom{2d-1}{d} \varphi\left(\frac{n}{d}\right).$$

Key words: Circulants; congruences; permanents.

1. Introduction

In 1952 M. Hall, Jr. proved the following theorem [1] (see footnote 1): If G is a finite abelian group of order n with elements $a_1, a_2, \ldots, a_n$, and $c_1, c_2, \ldots, c_n$ are n (not necessarily distinct) elements of G, then there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that the differences $a_{\sigma(1)} - a_1, a_{\sigma(2)} - a_2, \ldots, a_{\sigma(n)} - a_n$ are $c_1, c_2, \ldots, c_n$ in some order, if and only if

$$\sum_{i=1}^{n} c_i = 0. \tag{1}$$

The necessity of (1) is trivial, and Hall gives an elegant proof that condition (1) implies the existence of such a permutation σ. If G is the cyclic group of order n, then Hall's theorem may be rephrased in terms of congruences as follows: Let $x_0, x_1, \ldots, x_{n-1}$ be n nonnegative integers with

$$\sum_{i=0}^{n-1} x_i = n.$$

Then there is a permutation σ of $\{1, 2, \ldots, n\}$ such that

$$\sigma(i) - i \equiv k \pmod{n}$$

has exactly x_k solutions in i, $1 \leq i \leq n$, for each $k = 0, 1 \ldots, n-1$ if and only if

$$0x_0 + 1x_1 + \ldots + (n-1)x_{n-1} \equiv 0 \pmod{n}. \tag{2}$$

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The purpose of this note is to count the number of solutions of (2) in nonnegative integers x_i with $\sum_{i=0}^{n-1} x_i = n$. An application to the permanent of a circulant is given.

2. Main Result

The motivation having been given, we may now state and prove our main result.

THEOREM: *Let* n *be a positive integer. Let* F(n) *be the number of n-tuples* $(x_0, x_1, \ldots, x_{n-1})$ *satisfying:*

$$x_i \geq 0, \ (i = 0, 1, \ldots, n-1),$$

$$\sum_{i=0}^{n-1} x_i = n,$$

$$\sum_{i=0}^{n-1} i x_i \equiv 0 \pmod{n}.$$

Then

$$F(n) = \frac{1}{n} \sum_{d|n} \binom{2d-1}{d} \varphi\left(\frac{n}{d}\right),$$

where the summation extends over all positive integers d *dividing* n, *and where* φ *is Euler's function.*

PROOF: The proof uses generating functions. Define

$$f_n(w, z) = [(1-z)(1-wz) \ \ldots \ (1-w^{n-1}z)]^{-1}.$$

Then

$$f_n(w, z) = \left(\sum_{k=0}^{\infty} z^k\right)\left(\sum_{k=0}^{\infty} w^k z^k\right) \ \ldots \ \left(\sum_{k=0}^{\infty} w^{k(n-1)} z^k\right),$$

and it is clear that $F(n)$ is the sum of the coefficients of $z^n w^{nt}$, $0 \leq t \leq n-1$, in $f_n(w, z)$. Write

$$f_n(w, z) = \sum_{k=0}^{\infty} B_k z^k \qquad (B_k = B_k(n, w)).$$

Then because

$$f_{n+1}(w, z) = \frac{f_n(w, z)}{1-w^n z},$$

and

$$f_n(w, wz) = [(1-wz)(1-w^2 z) \ \ldots \ (1-w^n z)]^{-1},$$

we obtain

$$f_n(w, wz) = (1-z) f_{n+1}(w, z) = \frac{1-z}{1-w^n z} f_n(w, z).$$

Thus

$$\sum_{k=0}^{\infty} B_k w^k z^k = \frac{1-z}{1-w^n z} \sum_{k=0}^{\infty} B_k z^k,$$

so that

$$\sum_{k=0}^{\infty} B_k w^k z^k - \sum_{k=0}^{\infty} B_k w^{n+k} z^{k+1} = \sum_{k=0}^{\infty} B_k z^k - \sum_{k=0}^{\infty} B_k z^{k+1}.$$

Hence for $k \geq 1$,

$$B_k w^k - B_{k-1} w^{n+k-1} = B_k - B_{k-1},$$

or

$$B_k = \frac{1-w^{n+k-1}}{1-w^k} B_{k-1} \qquad (k \geq 1).$$

Thus since $B_0=1$,

$$B_k=\prod_{r=1}^{k}\frac{1-w^{n+r-1}}{1-w^r}\qquad(k\geq 0),$$

an empty product being 1. Therefore

$$f_n(w,z)=\sum_{k=0}^{\infty}\left\{\prod_{r=1}^{k}\frac{1-w^{n+r-1}}{1-w^r}\right\}z^k,$$

and $F(n)$ is the sum of the coefficients of w^{nt}, $0\leq t\leq n-1$, in

$$g_n(w)=\prod_{r=1}^{n}\frac{1-w^{n+r-1}}{1-w^r}=\prod_{r=1}^{n-1}\frac{1-w^{n+r}}{1-w^r}.$$

Now, $g_n(w)$ is a polynomial in w of degree $\sum_{r=1}^{n-1}\{n+r-r\}=n(n-1)$, and has nonnegative coefficients (since $f_n(w,z)$ has nonnegative coefficients). Since

$$\sum_{\zeta:\zeta^n=1}\zeta^k=\begin{cases}n, & \text{if } n \text{ divides } k\\ 0, & \text{otherwise}\end{cases}$$

we have

$$nF(n)=\sum_{\zeta:\zeta^n=1}g_n(\zeta),$$

the summations extending over all nth roots of unity.

Suppose now that ζ is a primitive dth root of unity, where $d|n$. Since

$$\lim_{w\to\zeta}\frac{1-w^{n+r}}{1-w^r}=\begin{cases}\dfrac{n+r}{r}, & \text{if } d \text{ divides } r\\ 1, & \text{otherwise}\end{cases}$$

we have that

$$g_n(\zeta)=\prod_{\substack{1\leq r\leq n-1\\ r\equiv 0 \bmod d}}\frac{n+r}{r}=\prod_{s=1}^{\frac{n}{d}-1}\frac{n+sd}{sd}=\binom{2\frac{n}{d}-1}{\frac{n}{d}}.$$

Therefore, since there are $\varphi(d)$ nth roots of unity which are primitive dth roots of unity,

$$F(n)=\frac{1}{n}\sum_{d|n}\binom{2\frac{n}{d}-1}{\frac{n}{d}}\varphi(d)$$

$$=\frac{1}{n}\sum_{d|n}\binom{2d-1}{d}\varphi\left(\frac{n}{d}\right).$$

This proves the theorem.

3. An Application

Let $A=[a_{ij}]$ be an $n\times n$ matrix. If σ is a permutation of $\{1, 2, \ldots, n\}$ then

$$a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$$

is called a *diagonal product* of A. The *permanent* of A, denoted by per (A), is the sum of the diagonal products of A. Thus

$$\text{per } (A) = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdot \cdot \cdot a_{n\sigma(n)},$$

the summation extending over all permutations of $\{1, 2, \cdot \cdot \cdot, n\}$. Suppose A is the n by n circulant

$$\begin{bmatrix} a_0 & a_1 & \cdot \cdot \cdot & a_{n-1} \\ a_{n-1} & a_0 & \cdot \cdot \cdot & a_{n-2} \\ \cdot & \cdot & & \cdot \\ a_1 & a_2 & \cdot \cdot \cdot & a_0 \end{bmatrix}.$$

Then the diagonal product $a_{1\sigma(1)} a_{2\sigma(2)} \cdot \cdot \cdot a_{n\sigma(n)}$ equals $a_0^{x_0} a_1^{x_1} \cdot \cdot \cdot a_{n-1}^{x_{n-1}}$ where x_k is the number of integers i, $1 \leq i \leq n$, such that $\sigma(i) - i \equiv k \pmod{n}$.

By Hall's theorem, if $x_0, x_1, \cdot \cdot \cdot, x_{n-1}$ are integers satisfying the hypothesis of the theorem, then $a_0^{x_0} a_1^{x_1} \cdot \cdot \cdot a_{n-1}^{x_{n-1}}$ is a diagonal product of the circulant A. Thus we have the following corollary.

COROLLARY: *The number of formally distinct diagonal products of an* n *by* n *circulant is given by*

$$\frac{1}{n} \sum_{d|n} \binom{2d-1}{d} \varphi\left(\frac{n}{d}\right).$$

Some other results on the permanent of a circulant are given by the authors in the reference below.[2]

(Paper 74B1–315)

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Orthogonal Decompositions of Tensor Spaces*

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Let V be an n-dimensional vector space over the complex numbers. Let H be a subgroup of S_m, the symmetric group on $\{1, \ldots, m\}$, and let $W = \bigotimes_1^m V$ be the tensor product of V with itself m times. In this note we give an orthogonal direct sum decomposition of W in terms of the system of inequivalent irreducible characters of H.

AMS Subject Classifications: Primary, 1580; Secondary, 2080.

Key words: Irreducible character; symmetry class of tensors; symmetry operator; tensor product.

1. Introduction

Let V be an n-dimensional vector space over the field of complex numbers. Let $W = \bigotimes_1^m V$ be the tensor product of V with itself m times. For $\sigma \epsilon S_m$, the symmetric group on $\{1, \ldots, m\}$, define the permutation operator $P(\sigma) : W \to W$ by $P(\sigma)v_1 \otimes \ldots \otimes v_m = v_{\theta(1)} \otimes \ldots \otimes v_{\theta(m)}$, $v_1, \ldots, v_m \epsilon V$, where $\theta = \sigma^{-1}$. It is easy to check [1] that $P(\sigma)$ is linear and that $P(\sigma)P(\tau) = P(\sigma\tau)$, $\sigma, \tau \epsilon S_m$. Any linear combination $T = \sum_{\sigma \epsilon S_m} a(\sigma)P(\sigma)$ is called a *symmetry operator* and the range of T is called a *symmetry class* of tensors.

In [4],[1] Weyl expressed W as a direct sum of symmetry classes. The corresponding symmetry operators were determined from the idempotent generators of the minimal right ideals of the group ring of S_m. In this paper we obtain an orthogonal direct sum decomposition of W into symmetry classes with respect to the inner product defined below.

Let $(\ ,\ )$ be an inner product on V. Define an inner product on W by

$$(x_1 \otimes \ldots \otimes x_m, y_1 \otimes \ldots \otimes y_m) = \prod_{i=1}^{m} (x_i, y_i), \quad x_i, y_i \epsilon V. \tag{1}$$

THEOREM 1: *Let* H *be a subgroup of* S_m *of order* h. *Let* $\chi_1, \ldots, \chi_k$ *be the complete system of inequivalent irreducible characters on* H, *with* χ_1 *having degree* r_i, $i = 1, \ldots, k$. *Define* $T_{\chi_i} : W \to W$ by

$$T_{\chi_i} = \frac{r_i}{h} \sum_{\sigma \epsilon H} \chi_i(\sigma)P(\sigma), \qquad i = 1, \ldots, k.$$

Let $V_{\chi_i}^m(H)$ *be the range of* T_{χ_i}. *Then with respect to the inner product* (1), W *is the orthogonal direct sum of the symmetry classes* $V_{\chi_i}^m(H)$:

$$W = \perp_{i=1}^{k} V_{\chi_i}^m(H). \tag{2}$$

*This work was done (1968–1969) while the author was a National Academy of Sciences-National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C.

[1] Figures in brackets indicate the literature references at the end of this paper.

In section 3, we discuss some of the difficulties involved in constructing a suitable basis for $V_\chi^m(H)$. This problem has been dealt with [1] when χ is linear, but little has been done otherwise. We attempt to show the extent to which the methods of [1] will apply when χ is of higher degree.

2. Proof of Theorem 1

To establish (2), it suffices to show that

$$\text{a. } T_{\chi_i} \text{ is hermitian,}$$

$$\text{b. } T_{\chi_i} T_{\chi_j} = \delta_{ij} T_{\chi_i},$$

$$\text{c. } \sum_{i=1}^{k} T_{\chi_i} = \text{identity.} \tag{3}$$

If "*" denotes the adjoint with respect to the inner product (1), then $(P(\sigma))^* = P(\sigma^{-1})$. Since $\overline{\chi^i(\sigma)} = \chi^i(\sigma^{-1})$, $T_{\chi i}^* = T_{\chi i}$. We now compute

$$T_{\chi_i} T_{\chi_j} = \frac{r_i r_j}{h^2} \sum_{\sigma,\, \tau \in H} \chi_i(\sigma)\chi_j(\tau) P(\sigma\tau) = \frac{r_i r_j}{h^2} \sum_{\mu \in H} P(\mu) \sum_{\sigma \in H} \chi_i(\sigma)\chi_j(\sigma^{-1}\mu).$$

The orthogonality relations for characters [3, p. 16] now imply that

$$T_{\chi_i} T_{\chi_j} = \frac{r_i r_j}{h^2} \sum_{\mu \in H} P(\mu) \frac{\delta_{ij} h \chi_j(\mu)}{r_j} = \delta_{ij} T_{\chi_i}.$$

Let e be the identity in H. Then

$$\sum_{i=1}^{k} T_{\chi_i} = \sum_{i=1}^{k} \frac{r_i}{h} \sum_{\sigma \in H} \chi_i(\sigma) P(\sigma)$$

$$= \frac{1}{h} \sum_{\sigma \in H} P(\sigma) \sum_{i=1}^{k} \chi_i(e)\chi_i(\sigma).$$

Again, the orthogonality relations imply that

$$\sum_{i=1}^{k} T_{\chi_i} = \frac{1}{h} \sum_{\sigma \in H} P(\sigma)\delta_{e,\sigma} h$$

$$= P(e),$$

the identity transformation on W. This proves (3).

We note that Weyl's decomposition of W is orthogonal with respect to the above inner product only when $m = 2$.

3. Bases for Symmetry Classes

As above, H is a subgroup of S_m, χ is an irreducible character on H of degree r, and T_χ and $V_\chi^m(H)$ are the associated symmetry operator and symmetry class. If $x_1 \otimes \ldots \otimes x_m$ is a decomposable tensor in W, set

$$T_\chi x_1 \otimes \ldots \otimes x_m = x_1 * \ldots * xm.$$

The tensor $x_1 * \ldots * x_m$ is called a decomposable element of $V_\chi^m(H)$. Now if $v_1, \ldots, v_n$ is a basis of V, the set of n^m tensors $v_{\alpha_1} \otimes \ldots \otimes v_{\alpha_m}$, $1 \leq \alpha_i \leq n$ is a basis of $W = \bigotimes_1^m V$. Thus there is a basis

of $V_\chi^m(H)$ consisting of decomposable elements. In [1], Marcus and Minc give a construction for such a basis when χ is linear, i.e., $r=1$. Let $\Gamma_{m,n}$ be the set of n^m sequences $\alpha=(\alpha_1, \ldots, \alpha_m)$, $1 \leq \alpha_i \leq n$. We write $\alpha \sim \beta$ if $\alpha^\sigma=(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(m)})=\beta$ for some $\sigma \epsilon H$. Clearly "$\sim$" is an equivalence relation on $\Gamma_{m,n}$. Choose lexicographically the lowest representative from each class determined by $\sim$. Call this set of representatives Δ. For each $\alpha \epsilon \Delta$, let H_α be the subgroup of all $\sigma \epsilon H$ such that $\alpha^\sigma=\alpha$. Let $\bar{\Delta}$ be the set of all $\alpha \epsilon \Delta$ such that $\sum_{\sigma \epsilon H_\alpha} \chi(\sigma) \neq 0$. If $v_1, \ldots, v_l$ is a basis of V, then the tensors

$$v_\alpha^* = v_{a_1} * \ldots * v_{a_m}, \quad \alpha \epsilon \bar{\Delta} \tag{4}$$

are a linearly independent set in $V_\chi^m(H)$. Moreover, if χ is linear, the tensors (4) are a basis of $V_\chi^m(H)$. (For details, see [1].) Note that $\bar{\Delta}$ depends only on H and χ, and if χ is linear $v_{\alpha\sigma}^* = \chi(\sigma) v_\alpha^*$ for all $\sigma \epsilon H$, $\alpha \epsilon \bar{\Delta}$ (see [1]). Thus in the linear case one can conveniently determine matrix representations of certain linear transformations on $V_\chi^m(H)$. Using these properties, Marcus, Minc and Newman (see, e.g., [1], [2]) have been able to prove a large class of inequalities for determinants, permanents, and other multilinear matrix functions. Thus it would be useful to find a basis of $V_\chi^m(H)$ when degree $\chi > 1$. The following result demonstrates some obstacles.

THEOREM 2: *If* $\mathrm{v}_1, \ldots, \mathrm{v}_n$ *is a basis of* V, *the tensors*

$$\mathrm{v}_\alpha^*, \quad \alpha \epsilon \bar{\Delta} \tag{5}$$

are a basis of $\mathrm{V}_\chi^m(\mathrm{H})$ *if and only if* χ *is linear. Moreover, if* $|\bar{\Delta}|$ *is the cardinality of* $\bar{\Delta}$ *then*

$$\dim V_\chi^m(H) \geq 2|\bar{\Delta}| \tag{6}$$

if χ *is not linear.*

PROOF. We may assume $v_1, \ldots, v_n$ is an orthonormal basis of V. The procedure in [1] still applies to show that the tensors (5) are linearly independent. For $\sigma, \tau \epsilon H$ and $\alpha, \beta \epsilon \bar{\Delta}$, we compute

$$\begin{aligned}(v_{\alpha\sigma}^*, v_{\beta\tau}^*) &= (T_\chi v_{a_{\sigma^{-1}(1)}} \otimes \ldots \otimes v_{a_{\sigma^{-1}(m)}}, T_\chi v_{\beta_{\tau^{-1}(1)}} \otimes \ldots \otimes v_{\beta_{\tau^{-1}(m)}}) \\ &= (T_\chi v_{a_{\sigma(1)}} \otimes \ldots \otimes v_{a_{\sigma(m)}}, v_{\beta_{\tau(1)}} \otimes \ldots \otimes v_{\beta_{\tau(m)}}),\end{aligned}$$

because T_χ is idempotent hermitian. Hence

$$\begin{aligned}(v_{\alpha\sigma}^*, v_{\beta\tau}^*) &= \frac{r}{h} \sum_{\rho \epsilon H} \chi(\rho) \prod_{t=1}^m (v_{a_{\sigma\rho^{-1}(t)}}, v_{\beta_{\tau(t)}}) \\ &= \frac{r}{h} \sum_{\rho \epsilon H} \chi(\rho) \prod_{t=1}^m (v_{a_{\sigma\rho^{-1}\tau^{-1}(t)}}, v_{\beta_t}).\end{aligned} \tag{7}$$

Since $\alpha, \beta \epsilon \bar{\Delta}$, the product in (7) is zero unless $\alpha=\beta$ and $\sigma\rho^{-1}\tau^{-1} \epsilon H_\alpha$.

Thus

$$\begin{aligned}(v_{\alpha\sigma}^*, v_{\beta\tau}^*) &= \delta_{\alpha,\beta} \frac{r}{h} \sum_{\sigma\rho^{-1}\tau^{-1} \epsilon H_\alpha} \chi(\rho) \\ &= \delta_{\alpha,\beta} \frac{r}{h} \sum_{\mu \epsilon H_\alpha} \chi(\sigma\tau^{-1}\mu).\end{aligned} \tag{8}$$

Thus we are finished if we can show that for each $\alpha \in \bar{\Delta}$ there is a $\sigma \in H$ such that v_α^* and $v_{\alpha\sigma}^*$ are linearly independent. Suppose this were false for some $\alpha \in \bar{\Delta}$. Then

$$v_{\alpha\sigma}^* = \eta(\sigma) v_\alpha^*, \ \sigma \in H, \tag{9}$$

where $\eta(\sigma)$ is a scalar. From (8) and (9).

$$\begin{aligned} \eta(\sigma)\,\overline{\eta(\tau)}(v_\alpha^*,\ v_\alpha^*) &= (v_{\alpha\sigma}^*, v_{\alpha\tau}^*) \\ &= \frac{r}{h} \sum_{\mu \in H_\alpha} \chi(\sigma\tau^{-1}\mu) \\ &= (v_{\alpha\sigma\tau^{-1}}^*, v_\alpha^*) \\ &= \eta(\sigma\tau^{-1})(v_\alpha^*, v_\alpha^*). \end{aligned} \tag{10}$$

Since $v_\alpha^* \neq 0$, (10) implies that η is a character on H of degree 1. Thus for all $\sigma \in H$

$$\begin{aligned} \sum_{\mu \in H_\alpha} \chi(\sigma\mu) &= \frac{h}{r}(v_{\alpha\sigma}^*, v_\alpha^*) \\ &= \frac{h}{r}\eta(\sigma)(v_\alpha^*, v_\alpha^*) \\ &= \eta(\sigma) \sum_{\mu \in H_\alpha} \chi(u). \end{aligned} \tag{11}$$

Multiply both sides of (11) by $\eta(\sigma^{-1})$ and sum on σ, obtaining

$$\begin{aligned} 0 \neq \sum_{\sigma \in H} \sum_{\mu \in H_\alpha} \chi(\mu) \\ = \sum_{\sigma \in H} \eta(\sigma^{-1}) \sum_{\mu \in H_\alpha} \chi(\sigma\mu) \\ = \sum_{\mu \in H_\alpha} \sum_{\sigma \in H} \eta(\sigma^{-1})\chi(\sigma\mu). \end{aligned} \tag{12}$$

As long as the degree of χ is greater than 1, the orthogonality relations imply that the right side of (12) is zero. This establishes (6) and thus Theorem 2 is proved.

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(Paper 74B1–316)

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Partitioned Hermitian Matrices*

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A class of Cauchy-Schwarz type inequalities for partitioned hermitian matrices is presented.

Key words: Generalized matrix function; positive semi-definite hermitian matrix.

Let M_n^r denote the set of $r \times n$ matrices over the complex numbers. Write M_n for M_n^n. Let G be a subgroup of S_{ml}, the symmetric group on ml symbols. Suppose λ is a character of degree 1 on G. If $X = (x_{ij}) \in M_{ml}$, the *generalized matrix function* of X is

$$d(X) = \sum_{\sigma \in G} \lambda(\sigma) \prod_{t=1}^{ml} x_{t\sigma(t)}.$$

Let $f : M_p \to M_m$ be any function. Then f induces a function $\varphi_f : M_{pq} \to M_{mq}$ as follows:

$$\varphi_f(X) = (f(X_{st}))$$

where $X = (X_{st})$ is a block matrix in which X_{st} is a $p \times p$ submatrix of X, $1 \leq s, t \leq q$.

Let $A_1, \ldots, A_k \in M_{pl}^r$. Let H be the block matrix

$$H = (A_i^* A_j) \in M_{pkl}.$$

(A_i^* is the conjugate transpose of A_i.)

LEMMA. *The matrix* H *is positive semi-definite hermitian* (psdh).

PROOF:

$$H = \begin{bmatrix} A_1^* & & 0 \\ & \ddots & \\ 0 & & A_k^* \end{bmatrix} \begin{bmatrix} I_r & \ldots & I_r \\ \vdots & & \vdots \\ I_r & \ldots & I_r \end{bmatrix} \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix},$$

where $I_r \in M_r$ is the identity matrix.

(Observe that in general the block matrix whose i, jth block is $A_j^* A_i$ is not *psdh*. Take

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, A_2 = I_2.)$$

Suppose f is such that $\varphi_f(H)$ is *psdh*. We may write

$$\varphi_f(H) = (\varphi_f(A_i^* A_j)).$$

*This work was done (1969–1970) while the author was a National Academy of Sciences-National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C. 20234.

Now, by a trivial consequence of a result of Marcus and Katz [4, Theorem 3],[1] the block matrix

$$(d(\varphi_f(A_i^* A_j)))$$

is *psdh*. Taking the determinant of the leading 2×2 principal submatrix, one obtains the main result:

THEOREM.

$$d\varphi_f(A_1^* A_1)\ d\varphi_f(A_2^* A_2) \geq d\varphi_f(A_1^* A_2)\ d\varphi_f(A_2^* A_1).$$

One easily sees that a string of further inequalities is available by taking any generalized matrix function of any of the leading principal submatrices.

Some functions, f, which have the property that φ_f sends psdh matrices to psdh matrices have been discovered. A partial list follows. Let $X \in M_p$.

1. Let $f(X) = X$. Then the theorem reduces to the well known result ([9, p. 168] or [5, p. 323])

$$d(A^*A)d(B^*B) \geq |d(A^*B)|^2.$$

When $p = l = 1$, this is the Cauchy-Schwarz inequality.

2. If $p = 1$, let $f(x) = x^r$, where r is a positive integer. Then Schur proved f has the required property [8]. Löwner [2] extended this to cover the case for any real number at least 1.

3. If $p = 1$, let $f(x) = |x|^2$ [3].

4. Let S be a subgroup of S_p and χ a character on S, let

$$f(X) = \sum_{\sigma \in S} \chi(\sigma) \prod_{t=1}^{p} x_{t\sigma(t)} \qquad [7].$$

5. For $l = 1$, $k = 2$, let $f(X) = \text{trace } (X^2)$, [6].

6. For a given symmetry class of tensors arising from a group and a character of degree 1, let $f(X) = K(X)$ be the associated matrix, or let $f(X)$ be any generalized matrix function of X [4, Theorem 3]. (For a discussion of these terms, see [9], [6], [10].)

7. Let $\alpha_1^2(X), \ldots, \alpha_p^2(X)$ be the squares of the singular values of X. Let

$$f(X) = \text{trace } K(\text{diag } (\alpha_1^2(X), \ldots, \alpha_p^2(X)))$$

be the Schur function of the squares of the singular values of the matrix X [6]. We may take

$$f(X) = E_s(\lambda_1(X), \ldots, \lambda_p(X))$$

to be the sth elementary symmetric function of the eigenvalues of X [1].

The author thanks Morris Newman for shortening the proof of the lemma.

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Bounds on a Polynomial*

T. J. Rivlin**

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Methods for computing the maximum and minimum of a polynomial with real coefficients in the interval [0, 1] are described, and certain bounds are given.

Key words: Bernstein polynomials; bounds; polynomials.

Introduction

Let $p(x)=a_0+a_1x+\ldots+a_nx^n$ have real coefficients. If I is the interval $[0, 1]$ and

$$M=\max_{x\in I} p(x), \qquad m=\min_{x\in I} p(x),$$

we wish to study some methods of approximating m and M relatively easily. Apart from their intrinsic interest, such methods would seem to have application in computations using interval arithmetic in which a basic operation is the determination of the range of a rational function with a given interval as domain. The restriction of our discussion to the interval I entails no loss of generality, since any interval $[a, b]$ can be mapped onto I by a linear function of x, and such changes of variable leave the set of polynomials of degree at most n invariant.

Some estimates of m and M were given by Cargo and Shisha.[1] They first observe that if $n\geq 1$ and $J=\{1, \ldots, k\}$,

$$\min_{j\in J} p\left(\frac{j}{k}\right)-\frac{1}{k}\sum_{j=1}^{n} j|a_j| \leq m \leq M \leq \max_{j\in J} p\left(\frac{j}{k}\right)+\frac{1}{k}\sum_{j=1}^{n} j|a_j|. \tag{1}$$

Then they note that p has a representation in Bernstein form, namely,

$$p(x)=\sum_{j=0}^{n} b_j\binom{n}{j}x^j(1-x)^{n-j}, \tag{2}$$

which leads immediately to the bounds

$$\min_{j\in N} b_j \leq m \leq M \leq \max_{j\in N} b_j, \tag{3}$$

where $N=\{0, \ldots, n\}$.

*An invited paper.
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[1] Cargo, G. T., and Shisha, O., The Bernstein form of a polynomial, J. Res. Nat. Bur. Stand. (U.S.), **70B** (Math. Sci.) No. 1, 79–81 (Jan–Mar. 1966).

The numbers $b_0, \ldots, b_n$ are determined by

$$b_j = \sum_{i=0}^{j} a_i \frac{\binom{j}{i}}{\binom{n}{i}}, \qquad j = 0, \ldots, n. \tag{4}$$

Our work is intended to be a sequel to the paper of Cargo and Shisha (see footnote 1). In section 1, we give a sharper result than (1). Section 2 is devoted to a generalization of the representation (2) and a subsequent improvement of (3), including estimates of the precision of the bounds obtained for m and M, which are due, essentially, to S. Bernstein.[2] The paper concludes with section 3, containing some results in the case that the coefficients $a_0, \ldots, a_n$ are complex numbers.

1. Estimates Using Function Values

THEOREM 1: *Suppose* $0 = t_0 < t_1 < \ldots < t_k = 1$, *and* $d_k = \max (t_{j+1} - t_j)$, $j = 0, \ldots, k-1$, *then, if* $K = \{0, \ldots, k\}$

$$\min_{i \in K} p(t_i) - \frac{d_k^2}{8} \max_{x \in I} |p''(x)| \leq m \leq M \leq \max_{i \in K} p(t_i) + \frac{d_k^2}{8} \max_{x \in I} |p''(x)|. \tag{5}$$

PROOF: Suppose $p(\xi) = M$ and

$$|t_j - \xi| \leq |t_i - \xi|, \; i = 0, \ldots, k,$$

so that

$$|t_j - \xi| \leq \frac{d_k}{2}.$$

According to Taylor's formula, we have

$$p(t_j) = p(\xi) + (t_j - \xi) p'(\xi) + \frac{(t_j - \xi)^2}{2} p''(\eta) \tag{6}$$

where $\eta \epsilon I$. If $\xi = 0$, 1, the right-most inequality in (5) is trivially true. If $0 < \xi < 1$ then $p'(\xi) = 0$ and (6) implies that

$$M \leq \max_{j \epsilon K} p(t_j) + \frac{(t_j - \xi)^2}{2} \max_{\eta \epsilon I} |p''(\eta)| \leq \max_{j \epsilon K} p(t_j) + \frac{d_k^2}{8} \max_{\eta \epsilon I} |p''(\eta)|.$$

An entirely analogous argument establishes the lower bound on m in (5).

COROLLARY: *If* $t_i = i/k$ *then* $d_k = 1/k$, *and since*

$$\max_{x \in I} |p''(x)| \leq \sum_{j=0}^{n} (j-1) j |a_j|$$

(5) *now becomes,*

$$\min_{j \in K} p\left(\frac{j}{k}\right) - \frac{1}{8k^2} \sum_{j=0}^{n} (j-1) j |a_j| \leq m \leq M \leq \max_{j \in K} p\left(\frac{j}{k}\right) + \frac{1}{8k^2} \sum_{j=1}^{n} (j-1) j |a_j|. \tag{7}$$

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[2] Bernstein, S. N., Collected Works, Vol. I, Translation: AEC-tr-3460, pp. 68–73.

The improvement of the error estimate in (7) over that in (1) is evident.

If we consider $p(x)=x-x^2$ then $M=1/4$, which is precisely the upper bound given by (7) with $k=1$.

2. The Bernstein Form

Suppose $s=0, 1, \ldots, k$ then we have

LEMMA 1:

$$x^s=\sum_{j=s}^{k}\frac{\binom{j}{s}}{\binom{k}{s}}\binom{k}{j}x^j(1-x)^{k-j}.$$

PROOF:

$$x^s=x^s(x+(1-x))^{k-s}=\sum_{i=0}^{k-s}\binom{k-s}{i}x^{s+i}(1-x)^{k-s-i}$$

$$=\sum_{j=s}^{k}\binom{k-s}{j-s}x^j(1-x)^{k-j}=\sum_{j=s}^{k}\frac{\binom{j}{s}}{\binom{k}{s}}\binom{k}{j}x^j(1-x)^{k-j}.$$

Thus, the functions

$$\binom{k}{j}x^j(1-x)^{k-j},\ j=0, \ldots, k$$

span the space, P_k, of polynomials of degree at most k and are linearly independent. Hence, if $p(x)=a_0+\ldots+a_nx^n$ (a_j are arbitrary numbers) and $k\geq n$ then

$$p(x)=\sum_{j=0}^{k}b_j^{(k)}\binom{k}{j}x^j(1-x)^{k-j} \tag{8}$$

where

$$b_j^{(k)}=\sum_{s=0}^{j}a_s\frac{\binom{j}{s}}{\binom{k}{s}},\ j=0, \ldots, k, \tag{9}$$

with the assumption that $a_s=0$ for $s>n$.

We call (8) the generalized Bernstein form of p.

THEOREM 2: *If* $p(x)=a_0+\ldots+a_nx^n$ *has real coefficients then for each* $k\geq n$ *we have (putting* $K=\{0, \ldots, k\}$)

$$b(k)=\min_{j\in K}b_j^{(k)}\leq m\leq M\leq\max_{j\in K}b_j^{(k)}=B(k).$$

PROOF: In view of (8) for each $x\epsilon I$, $p(x)$ is a convex combination of $b_0^{(k)}, \ldots, b_k^{(k)}$.

Following S. Bernstein, we can bound the discrepancies $B(k)-M$ and $m-b(k)$ as follows. We recall that if $f(x)$ is defined on I

$$B_k(f;x)=\sum_{j=0}^{k}f\left(\frac{j}{k}\right)\binom{k}{j}x^j(1-x)^{k-j}.$$

Suppose $0 \leq s \leq n$. Then $B_k(x^s; x) - x^s \epsilon P_k$ and hence

$$B_k(x^s; x) - x^s = \sum_{j=0}^{k} \delta_j(s) \binom{k}{j} x^j (1-x)^{k-j}.$$

THEOREM 3: *If* $k \geq n \geq 1$

$$\delta_j(s) \leq \frac{(s-1)^2}{k}; \; j=0, \ldots, k; \; s=0, \ldots, n. \tag{10}$$

PROOF. Since $B_k(1; x) \equiv 1$ and $B_k(x; x) \equiv x$ we have

$$\delta_j(0) = \delta_j(1) = 0, \; j=0, \; \ldots, \; k.$$

Henceforth, we assume $s \geq 2$.

(i) $0 \leq j < s$.

In view of Lemma 1, we have

$$\delta_j(s) = \left(\frac{j}{k}\right)^s \leq \left(\frac{s-1}{k}\right)^s \leq \left(\frac{s-1}{k}\right)^2 \leq \frac{(s-1)^2}{k}.$$

(ii) $2 \leq s \leq j$

$$\delta_j(s) = \left(\frac{j}{k}\right)^s - \frac{j!(k-s)!}{(j-s)!k!} = \left(\frac{j}{k}\right)^s - \frac{j(j-1) \; \ldots \; (j-(s-1))}{k(k-1) \; \ldots \; (k-(s-1))}$$

$$= \left(\frac{j}{k}\right)^s \left[1 - \frac{\left(1-\frac{1}{j}\right) \ldots \left(1-\frac{s-1}{j}\right)}{\left(1-\frac{1}{k}\right) \ldots \left(1-\frac{s-1}{k}\right)}\right]$$

$$\leq \left(\frac{j}{k}\right)^s \left[1 - \left(1-\frac{1}{j}\right) \ldots \left(1-\frac{s-1}{j}\right)\right]$$

$$\leq \left(\frac{j}{k}\right)^s \left[1 - \left(1-\frac{s-1}{j}\right)^{s-1}\right].$$

Applying the mean value theorem to $(1-x)^{s-1}$, we obtain

$$1 - \left(1-\frac{s-1}{j}\right)^{s-1} \leq \frac{(s-1)^2}{j},$$

hence

$$\delta_j(s) \leq \frac{(s-1)^2}{k} \left(\frac{j}{k}\right)^{s-1} \leq \frac{(s-1)^2}{k}. \tag{11}$$

REMARK: If $k \geq 2$, (10) can be improved slightly to

$$\delta_j(s) \leq \left(1-\frac{1}{k}\right) \frac{(s-1)^2}{k}, \tag{12}$$

since:

(i) If $j < s$,

$$\delta_j(s) \leqslant \frac{(s-1)^2}{k} \cdot \frac{1}{k} \leqslant \left(1-\frac{1}{k}\right)\frac{(s-1)^2}{k}.$$

(ii) If $s \leqslant j \leqslant k$,

$$\delta_k(s) = 0,$$

and so (11) can be restricted to $j < k$ leading to

$$\delta_j(s) \leqslant \frac{(s-1)^2}{k} \cdot \left(\frac{k-1}{k}\right)^s \leqslant \frac{(s-1)^2}{k} \cdot \left(\frac{k-1}{k}\right).$$

THEOREM 4: *If* $p(x) = a_0 + \ldots + a_n x^n$, *and* $k \geqslant n \geqslant 1$ *then for* $j = 0, \ldots, k$

$$p\left(\frac{j}{k}\right) - b_j^{(k)} = \delta_j \tag{13}$$

satisfies

$$|\delta_j| \leqslant \frac{A}{k} \tag{14}$$

where

$$A = \sum_{s=2}^{n} (s-1)^2 |a_s|. \tag{15}$$

If $k \geqslant 2$, *then*

$$|\delta_j| \leqslant A\,\frac{k-1}{k^2}, \; j = 0, \ldots, k.$$

PROOF: Since $B_k(p; x) - p(x) = \sum_{j=0}^{k} \delta_j \binom{k}{j} x^j (1-x)^{k-j}$, while in view of Theorem 3

$$B_k(p; x) - p(x) = \sum_{j=0}^{k} \sum_{s=0}^{k} a_s \delta_j(s) \binom{k}{j} x^j (1-x)^{k-j},$$

then

$$\delta_j = \sum_{s=0}^{k} a_s \delta_j(s),$$

and the Theorem follows from (10) and (12).

COROLLARY: *If* $a_0, \ldots, a_n$ are real, then

$$B(k) - M \leqslant \frac{A}{k}; \; m - b(k) \leqslant \frac{A}{k}. \tag{16}$$

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If $k \geq 2$

$$B(k) - M \leq A\frac{k-1}{k^2};\ m - b(k) \leq A\frac{k-1}{k^2}. \tag{17}$$

Thus, $B(k)$ converges to M as $k \to \infty$, and to determine how large we need to choose k in order to be a given distance away from M, we need only consult (16) or (17) (analogously for $b(k)$ and m). Cargo and Shisha (see footnote 1) give a difference table method for calculating $b_1^{(n)}, \ldots, b_{n-1}^{(n)}$ which can be used equally well to find $b_1^{(k)}, \ldots, b_{k-1}^{(k)}$ ($b_0^{(k)} = a_0$; $b_k^{(k)} = a_0 + \ldots + a_n$, always).

REMARK: Once a k has been chosen and the numbers $b_j^{(k)}$ ordered according to size, it may be possible to improve on $B(k)$ as an estimate for M, by means of a final correction. Suppose

$$B(k) = b_i^{(k)},\ i \neq 0, k,$$

and

$$\max_{j \neq i} b_j^{(k)} = b_l^{(k)} < b_i^{(k)}.$$

Put

$$p_j(x) = \binom{k}{j} x^j (1-x)^{k-j},\ j = 0, \ldots, k$$

then $0 \leq p_i(x) < 1$ for all $x \epsilon I$, and

$$p(x) = p_i(x) b_i^{(k)} + (1 - p_i(x)) \sum_{\substack{j=0 \\ j \neq i}}^{k} b_j^{(k)} \binom{k}{j} \frac{x^j (1-x)^{k-j}}{1 - p_i(x)}. \tag{18}$$

Since

$$\frac{p_j(x)}{1 - p_i(x)} \geq 0$$

and

$$\sum_{\substack{j=0 \\ j \neq i}}^{k} \frac{p_j(x)}{1 - p_i(x)} = 1,$$

we conclude from (18) that

$$p(x) \leq p_i(x) b_l^{(k)} + (1 - p_i(x)) b_l^{(k)} = (b_i^{(k)} - b_l^{(k)}) p_i(x) + b_l^{(k)}.$$

Since $p_i(x)$ assumes its maximum on I at $x = i/k$, we obtain, finally,

$$M \leq p_i\left(\frac{i}{k}\right) b_l^{(k)} + \left(1 - p_i\left(\frac{i}{k}\right)\right) b_l^{(k)} < B(k).$$

An analogous result holds for m.

3. The Complex Case

Suppose the coefficients $a_0, \ldots, a_n$ are complex numbers. Let C be the convex hull of $p(I)=\{p(x) : x \epsilon I\}$, and let C_k be the convex hull of $\{b_0^{(k)}, \ldots, b_k^{(k)}\}$. Then in view of (8), we have, for each $k \geq n$

$$C \subseteq C_k.$$

We have

LEMMA 2: *For each* $k \geq n$

$$C_{k+1} \subseteq C_k.$$

PROOF: It is an easy consequence of (9) that $b_0^{(k)}=a_0$, $b_k^{(k)}=a_0+\ldots+a_n$ and

$$b_j^{(k+1)}=\left(1-\frac{j}{k+1}\right) b_j^{(k)}+\frac{j}{k+1} b_{j-1}^{(k)}; \; j=1, \ldots, k.$$

Thus

$$b_j^{(k+1)} \epsilon C_k, \; j=0, \ldots, k+1$$

and the lemma follows.

Note that C and C_k, $k=n, n+1, \ldots$ are each compact sets in the plane, and that

$$C \subseteq \bigcap_{k=n}^{\infty} C_k. \tag{19}$$

Indeed,

THEOREM 5:

$$C=\bigcap_{k=n}^{\infty} C_k.$$

PROOF. Suppose there exists

$$z \epsilon \bigcap_{k=n}^{\infty} C_k$$

and $z \notin C$. Since C is compact, dist $(z, C)=d>0$. Choose $k \geq n$ so large that $(A/k)<d$. Now, there exist $\lambda_0, \ldots, \lambda_k$ satisfying $\lambda_j \geq 0$, $\Sigma \lambda_j=1$ such that

$$z=\sum_{j=0}^{k} \lambda_j b_j^{(k)},$$

hence, in view of (13) and (14)

$$\sum_{j=0}^{k} \lambda_j p\left(\frac{j}{k}\right)-z=\sum_{j=0}^{k} \lambda_j \delta_j,$$

and

$$\left|\sum_{j=0}^{k} \lambda_j p\left(\frac{j}{k}\right)-z\right| \leq \frac{A}{k}<d. \tag{20}$$

But

$$\sum_{j=0}^{k} \lambda_j p\left(\frac{j}{k}\right) \in C,$$

and (20) contradicts dist $(z, C) = d$, thus proving the theorem.

Note that the same proof shows that

$$\text{dist } (C_k, C) \leq \frac{A}{k}.$$

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Entire Functions of Exponential Type *

Fred Gross **

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Let f be an entire function satisfying for some integer p and some constant C

$$\sum_{j=0}^{N}\left(\int_0^{2\pi}\left/ f^{(j)}(re^{i\theta})/^p d\theta\right.\right)^{1/p} \geq C\sum_{j=N+1}^{\infty}\left(\int_0^{2\pi}\left/ f^{(j)}(re^{i\theta})\right/^p\right)^{1/p} \tag{1}$$

for sufficiently large r. Then f is of exponential type. Conversely, (1) is satisfied whenever f is periodic of exponential type. Similar conditions on the maximum moduli $M_f(j)(r)$ yield the same result. The analogous condition on $/f^{(j)}/$ is also discussed.

Key words: Bounded index; convexity; entire function; exponential type; maximum modulus.

1. Introduction

An entire function, $f(z)$, has a Taylor expansion about any point a in the complex plane of the form

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$$

Since this series is absolutely convergent everywhere in the plane, $|a_n|$ must approach zero as n approaches infinity. Consequently, there exists for each a, an index $n(a)$ for which $|a_n|$ is a maximal coefficient. B. Lepson [3][1] raised the question of characterizing entire functions for which $n(a)$ is bounded in a.[2] In the sequel we shall consider certain interesting variations of Lepson's problem. Though some of the results of this paper can also be obtained from the Wiman-Valiron theory, we shall use a more elementary and direct method which seems of interest in itself.

2. Preliminaries

We begin by defining the notion of bounded index and the related properties that we shall consider in the sequel.

DEFINITION 1: An entire function is said to be of bounded index if and only if, there exists an integer N, such that for all z

$$\max_{i=0,1,2,\ldots N}\{|f^{(i)}(z)|/i!\} > |f^{(j)}(z)|/j!; \tag{1}$$

$j=0, 1, 2, 3 \ldots$, where $f^{(0)}(z)$ denotes $f(z)$.

The bounded index condition is closely related to the condition that for some integer $N \geq 0$, some $C>0$,

$$\sum_{i=0}^{N}|f^{(i)}(z)|/i! \geq C|f^{(j)}(z)|/j!;\ j=N+1, N+2, \ldots \tag{2}$$

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[1] Figures in brackets indicate the literature references at the end of this paper.

[2] Since the completion of this paper a number of papers have appeared on this problem. e.g., [4, 5]

We concern ourselves with a slight variation of (2); namely that for some integer $N \geq 0$, some $C > 0$, some $r_0 > 0$ and all z with $|z| > r_0$,

$$\sum_{i=0}^{N} |f^{(i)}(z)|/i! > C \sum_{j=N+1}^{\infty} |f^{(j)}(z)|/j!. \tag{3}$$

We also consider those variations of (3) obtained by replacing $|f^{(i)}(z)|$ by

$$\left(\int_0^{2\pi} |f^{(i)}(re^{i\theta})|^p d\theta \right)^{1/p}$$

p any positive integer, and by $M_f(i)^{(r)}$, the maximum modulus of $f^{(i)}$ on $|z| = r$, respectively. In order to simplify notation we let $0(|f|, a, b)$, $0\left(\int_f, p, a, b \right)$ and $0(M_f, a, b)$ denote

$$\sum_{j=a}^{b} |f^{(j)}(z)|/j!, \quad \sum_{j=a}^{b} \frac{\left(\int_0^{2\pi} |f^{(j)}(re^{i\theta})|^p d\theta \right)^{1/p}}{j!}$$

and

$$\sum_{j=a}^{b} M_f^{(j)}(r)/j!$$

respectively.

We shall show in the sequel that entire functions satisfying condition (3) or any of the conditions obtained from it by the substitutions suggested above are functions of exponential type.

3. Entire Functions Satisfying (3) and Related Properties

THEOREM 1. *Let* f *be entire and* C *be a positive constant. If for* i = 0, 1, 2, . . ., N, f *satisfies one of the following for all* z *with* |z| *sufficiently large:*

$$\text{(a)}\ \ 0(|f^{(0)}|, 0, N) > C0(|f^{(0)}|, N+1, \infty)$$

$$\text{(b)}\ \ 0\left(\int_f (i), p, 0, N \right) > C0\left(\int_f (i), p, N+1, \infty \right)$$

for some positive integer p,

$$\text{(c)}\ \ 0(M_{f(i)}, 0, N) > C0(M_{f(i)}, N+1, \infty),$$

then f *is of exponential type.*

PROOF: For convenience we choose $C = 1$. The proof for other C is similar. For any arbitrary entire F and any complex number A, we have,

$$F(z) = \sum_{j=0}^{\infty} \frac{F^{(j)}(A)}{j!} (z-A)^j.$$

Let n be any integer, a and ξ complex numbers with $|\xi| = 1$. Choosing $A = (n-1)\xi + a$, $z = n\xi + a$ and $F = f^{(i)}$ we obtain

$$|f^{(i)}(a+n\xi)| \leq 0(|f^{(i)}(a+(n-1)\xi)|, 0, \infty), \tag{4}$$

for $i = 0, 1, 2, \ldots$.

Assume that hypothesis (a) holds. Then (4) yields

$$|f^{(i)}(a+n\xi)| \leq 2\,0(|f^{(i)}(a+(n-1)\xi)|,\ 0, N); \tag{5}$$
$$i=0, 1, 2, \ldots, N.$$

One observes that for $i \leq N$

$$\begin{aligned} 0(|f^{(i)}|, 0, N) &= \sum_{j=0}^{N} |f^{(i+j)}(z)|/j! \\ &= \sum_{j=0}^{N} \frac{|f^{(i+j)}(z)|}{(j+1)!}\,\frac{(j+i)!}{j!} \\ &\leq \sum_{l=0}^{2N} \frac{|f^{(l)}(z)|}{l!}\,(2N)^N \\ &\leq 2.\sum_{l=0}^{N} \frac{|f^{(l)}(z)|}{l!}\,(2N)^N = 2^{N+1}N^N 0(|f|, 0, N). \end{aligned}$$

Equations (5) and (6) yield

$$0(|f(a+n\xi)|, 0, N) \leq (N+1)2^{N+1}N^N 0\,(|f(a+(n-1)\xi)|, 0, N). \tag{7}$$

Letting $\lambda = (N+1)2^{N+1}N^N$ and using (7) recursively we have

$$0(|f(a+n\xi)|, 0, N) < \lambda^n 0(|f(a)|, 0, N). \tag{8}$$

For $|a| < 1$, we get

$$0(|f(a+n\xi)|, 0, N) < C\lambda^n,\ C \text{ constant}.$$

Letting $z = a+n\xi$, $r = |z|$ we get

$$|f(z)| < C\lambda^{2r}.$$

Hence, f must be of exponential type.
If (c) is assumed instead of (a) the argument is almost identical to the one above.
If (b) is assumed instead of (a), then letting $M_p(r)$ (or $M_p(r, f)$) denote

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p},$$

we have as before

$$M_p(r) < C\lambda^{2r}. \tag{9}$$

One readily sees, by means of Cauchy's formula that for $R > r > 0$

$$M_f(r) \leq \left(\frac{R}{R-r}\right)^{1/p} M_p(R)$$

Furthermore it is easy to verify that $M_p(R)$ (see Hardy [1]) is continuous and increases faster than any power of r whenever f is transcendental.

Letting $R=\frac{1}{\log M_p(R)}+r$ and applying a lemma of Bore [2 pp. 374–376], we get for some $\epsilon>0$ that

$$M_f(r) < (M_p(r))^{1+\epsilon}$$

outside a set of r of finite measure. One can easily show however that if there exists an infinite sequence r_n such that

$$M_f(r_n) > e^{C_n r_n},\ C_n \to \infty,$$

then there exists a set of r of infinite measure with the same property. Thus $M_f(r) < C\lambda^{2r}$ for sufficiently large r and our proof is complete.

4. The Converse of Theorem 1

In general the converse of Theorem 1 is not true, since one can easily find functions of order zero which do not satisfy (a). For example $f=\prod_{n=0}^{\infty}\left(1-\frac{z}{n^n}\right)^n$. We do, however, have the following related theorem.

THEOREM 2: *A periodic function of exponential type satisfies* (b) *and* (c) *for sufficiently large* r *and* i$=0, 1, 2, \ldots, N$.

PROOF: We prove (b). The proof for (c) is similar. Any such function g may be written as

$$g(z)=\sum_{j=1}^{n}\left[a_j(e^{ijz}+e^{-ijz})+b_j(e^{ijz}-e^{-ijz})\right]+C. \tag{10}$$

Without any loss of generality we may assume that $C=0$.
For $g^{(K)}$ we have:

$$g^{(K)}(z)=\sum_{j=1}^{n}(ij)^K[a_j(e^{ijz}+(-1)^K e^{-ijz})+b_j(e^{ijz}+(-1)^{K+1}e^{-ijz})] \tag{11}$$

Now $\log M_p(r)$ is a convex function of $\log r$ and $M_p(r)$ grows faster than any power of r (see Hardy [1]). Furthermore, one may assume without any loss of generality that $M_p(1)=1$. Using these facts one can easily verify by means of the three circles theorem, that for any constant C and any $\epsilon>0$

$$M_p((1+\epsilon)r) > CM_p(r), \tag{12}$$

for sufficiently large $r>r(C, p, f)$. Let $g_j=a_j(e^{ijz}+e^{-ijz})+b_j(e^{ijz}e^{-ijz}))$ and let $\bar{g}_1=g_n$.

$$\text{Then } M_p(r, g(z)) \geq M_p(r, g_n(z))-M_p\left(r, \sum_{j=1}^{n-1} g_j(z)\right).$$

Thus, by (12), for any C, $M_p(r, g_n(z)) > CM_p(r, g_j(z))$ and consequently for any $\delta>0$.

$$M_p(r, g(z)) > (1-\delta)M_p(r, g_n(z)) = (1-\delta)M_p(r, \bar{g}, (z))$$

for sufficiently large r.

Similarly (11) and (12) yield

$$M_p(r, g^{(j)}(z)) \geq \frac{n^j}{j!}(1-\delta)M_p(r, \bar{g}_1)$$

for j even and for j odd

$$M_p(r, g^{(j)}(z)) \geq \frac{n^j}{j!}(1-\delta)M_p(r, \bar{g}_2),$$

where $$\bar{g}_2 = a_n(e^{inz} - e^{-inz}) + b_n(e^{inz} + e^{-inz}).$$

Using these inequalities, one computes for any integer N, which we may assume to be even without any loss of generality,

$$\begin{aligned} 0\left(\int_g, p, 0, N\right) &\geq (i-\delta)\left(1+\frac{n^2}{2!}+\frac{n^4}{4!}+\ldots+\frac{n^N}{N!}\right)\left(\int_0^{2\pi}|\bar{g}_1(z)|^p d\theta\right)^{1/p} \\ &+(1-\delta)\left(n+\frac{n^3}{3!}+\frac{n^5}{5!}+\ldots+\frac{n^{N-1}}{(N-1)!}\right)\left(\int_0^{2\pi}|\bar{g}_2(z)|^p d\theta\right)^{1/p} \end{aligned} \tag{13}$$

for sufficiently large $|z| = r$.

On the other hand (12) also yields

$$\begin{aligned} 0\left(\int_g, p, 0, \infty\right) &\leq (1+\delta)\left(\frac{n^{N+2}}{(N+2)!}+\frac{n^{N+3}}{(N+3)!}+\ldots\right)\left(\int_0^{2\pi}|\bar{g}_1|^p d\theta\right)^{1/p} \\ &+(1+\delta)\left(\frac{n^{N+1}}{(N+1)!}+\frac{n^{N+3}}{(N+3)!}+\ldots\right)\left(\int_0^{2\pi}|\bar{g}_2|^p d\theta\right)^{1/p} \end{aligned}$$

for sufficiently large $|z| = r$.

Thus, choosing N sufficiently larger than n our conclusion follows for g. It is clear that it follows for $g^{(i)}$, $i = 1, 2, \ldots$, as well with N fixed.

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Publications of the National Bureau of Standards*

Selected Abstracts

Lehner, J., Newman, M., **On Reimann surfaces with maximal automorphism groups,** *Glasgow Math. J.* **8,** *Part 2, 102–112 (July 1967).*

Key words: Automorphisms; Fuchsian groups; modular group; Riemann surfaces.

Families of Fuchsian groups with associated Riemann surfaces having maximal automorphism groups are constructed, by a method which goes from the non-compact case to the compact case.

Marcus, M., Newman, M., **Doubly stochastic associated matrices,** *Duke Math. J.* **34,** *No. 3, 591–598 (Sept. 1967).*

Key words: Associated matrices; compounds; doubly stochastic matrices; induced powers, tensor products.

It is proved that the associated matrix $K(A)$ is doubly stochastic if and only if A is a scalar multiple of a permutation matrix, unless $K(A)$ is a Kronocker power. In this case $K(A)$ is doubly stochastic if and only if A is a scalar multiple of a doubly stochastic matrix.

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Key words: Automorphisms; cyclotomic fields; diophantine equations; roots of unity.

An effective method for determining all rational solutions $x_1, x_2, \ldots, x_t$ of the equation $\sin \pi x_1 \sin \pi x_2 \ldots \sin \pi x_t =$ positive rational is given.

Newman, M., **Some results on roots of unity, with an application to a diophantine problem,** *Aequat. Math.* **2,** *No. 2/3, 163–166 (Nov. 28, 1967).*

Key words: Cyclotomic fields; diophantine equations; multiplicative functions.

The principal result of this note is the determination of all rationals x, y such that $\sin \pi x \sin \pi y$ is rational.

Other NBS Publications

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Formulation of a nonanalytic equation of state for parahydrogen. R. D. Goodwin.

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Odd configurations in singly-ionized copper. C. Roth.

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Phase transformation in barium tetraborate. C. R. Robbins and E. M. Levin.

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Standardization of cesium-137 gamma-ray sources in terms of exposure units (Roentgens). T. P. Loftus.

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Blemish formation in processed microfilm. II. C. I. Pope.

Positive and negative ion sublimation from transition metal surfaces: A review of some recent results. M. D. Scheer.

Generation of controlled low pressures of nitrogen by means of dissociation equilibria. R. H. Orcutt.

Temperature coefficient of the bismuth I–II transition pressure. J. C. Houck.

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